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Combinatorics of triangular recurrence sequences

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Abstract- Resumé

Abstract

In this thesis, we explore three complementary approaches for studying *triangular recurrence sequences*: **formal grammars**, **weighted paths**, and **Riordan matrices**. Our goal is to derive explicit formulas, establish new identities, and compare these results with existing findings in the literature. Using the **formal grammar approach**, we model the sequences through *systems of differential equations* to obtain explicit expressions for the coefficients $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$. In the **weighted path approach**, we interpret $T(n,k)$ as the *total weight of combinatorial paths*, allowing us to study the coefficients $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ from a structural perspective, with explicit formula. Finally, the **Riordan matrix method** provides a powerful algebraic framework, which we use to derive explicit formulas in specific cases, such as when $b_2 = 0$, and to characterize certain families of triangular recurrences. Here, we also present the limit of the cases where the Riordan matrix cannot be used. This work illustrates how combining combinatorial, algebraic, and analytic techniques can yield deeper insights into the structure and properties of triangular recurrence sequences.

Keywords: triangular recurrence, weighted paths, Riordan matrix, formal grammar, combinatorial operators, differential equations.

Resumé

Dans ce mémoire, nous proposons trois approches pour étudier les suites à récurrence triangulaires: grammaires formelles, chemins pondérés et matrice de Riordan. Nous déduisons des identités et les confrontons avec les précédentes littératures. Pour la **méthode grammaticale**, nous utilisons les systèmes d'équations différentielles pour essayer de trouver la formule explicite de $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$. Pour les **chemins pondérés**, nous identifions $T(n,k)$ avec un poids total d'un chemin et étudier $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ à partir de ce chemins. Nous faisons aussi secourir la **matrice de Riordan**. Nous prouvons la formule explicite pour certains cas: $b_2 = 0$ et des matrices de Riordan pour certains cas. Ici, nous présentons aussi la limite des cas où l'on ne peut pas utiliser la matrice de Riordan. Ce travail illustre comment la combinaison de techniques combinatoires, algébriques et analytiques peut offrir une compréhension plus approfondie de la structure et des propriétés des suites récurrentes triangulaires.

Mots clés: Récurrence triangulaire, chemins ponderées, matrice de Riordan, grammaire formelle, opérateur différentiel et système des équation différentielle combinatoire.

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$c_{m,r}$	:	r-Whitney numbers of the first kind (unsigned).
$w_{m,r}$	:	r-Whitney numbers of the first kind (signed).
$W_{m,r}$	:	r-Whitney numbers of the second kind.
$S(n,k)$	:	Stirling numbers of the second kind.
$s(n,k)$	:	Stirling numbers of the first kind.
$\|A\|$	:	Cardinality of the set A.
$\#A$	:	Cardinality of the set A.
$\|A\|_v$	:	Total weight of A with respect to the v-weighting.
$(x)^{(\bar{k})}$	:	$x(x+1)\cdots(x+k-1)$.
$(x)^{(\underline{k})}$	:	$x(x-1)\cdots(x-k+1)$.
$(x\|a)^{(\bar{k})}$	:	$x(x+a)\cdots(x+(k-1)a)$.
$(x\|a)^{(\underline{k})}$	:	$x(x-a)\cdots(x-(k+1)a)$.
$\sum_n$	:	Sum over $n \in \mathbb{N}$.
$WL_{m,r}$	:	r-Lah numbers.
$\delta_{n,k}$	:	Kronecker delta: 1 if $n=k$ and 0 otherwise.
$\mathcal{M}_n(\mathbb{R})$	:	Set of $n \times n$ real matrices.
$\mathbb{I}_n$	:	Identity matrix of order n.
$\mathbb{I}$	:	Identity matrix of infinite order.
$[n]$	:	The set $\{1, 2, \cdots, n\}$.

Introduction

In mathematics, certain numbers satisfy a triangular recurrence of the form

$$T(n,k) = a_{n,k}T(n-1,k) + b_{n,k}T(n-1,k-1).$$

or

$$T(n,k) = (a_0 + a_1k + a_2n)T(n-1,k) + (b_0 + b_1k + b_2n)T(n-1,k-1). \tag{1}$$

After introducing the binomial coefficients, the Stirling numbers, and the Eulerian numbers, Graham et al. [1] proposed a generalization problem of the form (1). In combinatorics, several approaches have been developed concerning these types of numbers, such as the GKP numbers[1]. One of the results is due to Neuwirth [2], who obtained an explicit formula for the case $b_2 = 0$ using the Galton triangle. Spivey [3] identified several particular cases using finite differences. Analytical approaches have also been studied in the literature, notably by Théorêt [4], Wilf [5], and Barbero [6]. Cheon [7] treated the r-Whitney numbers using a more algebraic approach by studying Dowling lattices and Riordan matrices, and derived new identities for the r-Whitney numbers. Motivated by this perspective, we aim to generalize this idea: the use of Riordan matrices for several families of numbers of the form (1). Grammatical approaches and interpretations have also been developed by Hao and Zhou [8, 9]. We will also make use of their results through the grammar relation and differential equations (Randrianirina, Dumont [10, 11]) in order to better understand sequences of the form $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$.

The recurrence relation (1) generally arises from the enumeration of combinatorial structures and objects. Moreover, the associated grammar may also provide new combinatorial interpretations of the numbers $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$, as in the work of Ramirez [12] on the r-Whitney numbers.

On the other hand, this triangular recurrence (1) may be viewed as describing sequences of steps with weights. Thus, one objective of this thesis is to study such sequences as total weights of certain paths, together with usual initial conditions:

$$T(0,0) = 1, \quad T(n,k) = 0 \text{ if } n < \max(k,0). \tag{2}$$

We restrict ourselves to the setting where $a_{n,k} = a_0 + a_1k + a_2n$ and $b_{n,k} = b_0 + b_1k + b_2n$, with $a_{n,k} \neq 0$ and $b_{n,k} \neq 0$. In addition, it is easier to work when $b_{n,k} = 1$. This is the case we will consider in most of our study. However, generalizations of Eulerian numbers where $b_{n,k} \neq 1$ are also important when studying sequences satisfying (1).

This work begins with a review of the necessary background and grammatical approaches combined with systems of combinatorial and analytic differential equations. We will provide examples illustrating how this method can lead to explicit relations and combinatorial interpretations

[1] Graham, Knuth and Patashnik.

for certain cases. Next, we introduce an approach using weighted paths to obtain expressions for $T(n,k)$. We present a general formula for $T(n,k)$ which requires a certain amount of analysis. Examples involving known identities are given to provide additional insight into the theory. We also provide an alternative proof of the result of Neuwirth [2] in the case where $b_2 = 0$. The final part concerns the exponential Riordan matrix. We discuss the importance of this approach and also examine cases where it cannot be applied. Explicit formulas are provided for the case $b_{n,k} = 1$, for $b_2 = 0$, and for several other situations.

Chapter 1

Generalities and Grammatical Approach

We study sequences satisfying the relation (1) by using the correspondence between grammars, combinatorial and analytic differential equations. The idea is to use Theorem 1.5.1 and apply the grammar (1.19)

$$G_2 = \{u \to u^{b_1+b_2+1}v^{a_1+a_2};\ v \to u^{b_2}v^{a_2+1}\}.$$

We show that this grammar yields interpretations for certain sequences $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfying (1). In particular, the r-Whitney–Eulerian numbers and the cases where $b_{n,k} = 1$ are obtained explicitly. Since Theorem 1.5.1 is formulated in the language of the theory of species, we first recall some brief notions about species.

1.1 Preliminaries

In this section, we recall the notions required to read this work. These include weighted sets, paths in $\mathbb{N} \times \mathbb{N}$, and the theory of species together with the resolution of combinatorial differential equations.

1.1.1 On paths and weights

Définition 1.1.1 (Definition of a weighted set)**.** *A weighted set is a pair (E, v) where E is a set and v is a valuation, that is, a map from E to a ring $\mathbb{A}$[1]. The natural definition of a morphism of weighted sets is then a map that preserves weights, that is, $f : (E, v) \longrightarrow (F, w)$ is a morphism if*

$$\forall x \in E: \quad w(f(x)) = v(x).$$

Thus, we have the notion of isomorphism of weighted sets, which is nothing but a bijective morphism. Moreover, for a subset A of E, the weight of A is defined as the total weight of all its elements,

$$|A|_v := \sum_{x\in A} v(x).$$

[1]We only need addition and multiplication.

It is clear that if $f : (E, v) \longrightarrow (F, w)$ is an injective morphism and $A \subseteq E$, then $|f(A)|_w = |A|_v$. In particular, a classical fact about weighted sets is that an isomorphism preserves total weight.

Définition 1.1.2 (Path)**.** *A path in the lattice $\mathbb{N} \times \mathbb{N}$ is a sequence of steps $(p_i)_{i\in\mathbb{N}}$, $c = (p_0, p_1, \ldots, p_n)$, where the following steps are allowed:*

- *a North step from $(i, j-1)$ to (i, j), or a South step from $(i, j+1)$ to (i, j); or*
- *an East step from $(i-1, j)$ to (i, j), or a West step from $(i+1, j)$ to (i, j).*

We may also define North–East steps from $(i-1, j-1)$ to (i, j) and North–West steps. Furthermore, we define double steps such as North–North from $(i, j-2)$ to (i, j) and East–East similarly.

1.1.2 Some reminders on the theory of species

Définition 1.1.3. *A $\mathbb{B}$-species is an endofunctor $\mathcal{F}$ of $\mathbb{E}$[2]. The elements of $\mathcal{F}(X)$ are the $\mathcal{F}$-structures on X for each finite set X, and $\mathcal{F}(\sigma)$ is called the transport of structure along σ for each bijection σ.*
A weighted $\mathbb{L}$-species is a functor from the category of finite totally ordered sets with increasing bijections to the category of summable $\mathbb{A}$-weighted sets.

This definition is intended for readers familiar with the language of categories. It appears in Joyal [13]. Understanding it requires some basic knowledge of category theory (see MacLane [14] or Riehl 15). Otherwise, one can understand a $\mathbb{B}$-species without speaking too much about categories: a *species of structures* $\mathcal{F}$ associates:

- to each finite set A, a finite set $\mathcal{F}(A)$, called the *set of structures on* A;
- to each bijection $\sigma : A \to B$, a map

  $$\mathcal{F}(\sigma) : \mathcal{F}(A) \longrightarrow \mathcal{F}(B),$$

  called the *transport of structures along the bijection* σ, satisfying:

  1. $\mathcal{F}(\tau \circ \sigma) = \mathcal{F}(\tau) \circ \mathcal{F}(\sigma)$, for all bijections $\sigma : U \to V$, $\tau : V \to W$,
  2. $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$.

Exemple 1.1.1. *The following are the most common first examples. The species $\mathcal{S}$ of permutations assigns to each finite set E the set $\mathcal{S}(E) = \{$bijections $E \to E\}$. If $\sigma : E \to F$ is a bijection and $\tau \in \mathcal{S}(E)$, then $\mathcal{S}(\sigma)(\tau) = \sigma^{-1} \circ \tau \circ \sigma$. Similarly, the species $\mathcal{E}$ of sets assigns to each finite set E the singleton $\mathcal{E}(E) = \{E\}$. For any bijection $\sigma : E \to F$, we define $\mathcal{E}(\sigma)(E) = F$.*

Whereas an $\mathbb{L}$-species is a construction $\mathcal{F}_v$ which, to each finite totally ordered set l, associates a weighted set $\mathcal{F}_v[l]$, the set of $\mathcal{F}_v$-structures, and to the unique increasing bijection $\sigma : l \to l'$, associates a weight-preserving increasing bijection

$$\mathcal{F}[\sigma] : \mathcal{F}[l] \longrightarrow \mathcal{F}[l'],$$

[2] $\mathbb{E}$ is the category of finite sets with bijections.

the transport of structure, such that

$$\mathcal{F}[\sigma\tau] = \mathcal{F}[\sigma]\,\mathcal{F}[\tau] \quad \text{and} \quad \mathcal{F}[1_l] = 1_{\mathcal{F}[l]}.$$

For more information and examples, the reader is invited to consult Bergeron [16] and Randrianirina [10].

Remarque 1.1.1. Being a functor, a species preserves isomorphisms (bijections). Therefore, for a $\mathbb{B}$-species $\mathcal{F}$, the cardinality of $\mathcal{F}(X)$ depends only on the cardinality of X.

We also associate a generating series to every species $\mathcal{F}$:

$$\mathcal{F}(t) = \sum_{n\geq 0} |\mathcal{F}([n])| \frac{t^n}{n!},$$

where $|\mathcal{F}([n])|$ is either a cardinality or a total weight, depending on the type of species.

Définition 1.1.4 (10)**.** *A* mixed species *is a functor $\mathcal{F} : \mathbb{L} \times \mathbb{B} \to \mathbb{E}_A$, where $\mathbb{E}_A$ is the category of summable A-weighted sets. It is therefore a rule which:*

1. *For each pair (l, U), where l is a finite totally ordered set and U is an arbitrary finite set, associates a summable weighted set $\mathcal{F}_v[l, U]$;*
2. *For each pair $(\tau, \sigma) : (l, U) \to (l_0, U_0)$, where $\tau : l \tilde{\to} l_0$ is an increasing bijection and $\sigma : U \to U_0$ is any bijection, associates an isomorphism of weighted sets*

$$\mathcal{F}[\tau, \sigma] : \mathcal{F}[l, U] \to \mathcal{F}[l_0, U_0],$$

called the transport of structures, satisfying

$$\mathcal{F}[\tau, \sigma]\mathcal{F}[\tau', \sigma'] = \mathcal{F}[\tau\tau', \sigma\sigma'] \quad \textit{and} \quad \mathcal{F}[1_l, 1_U] = 1_{\mathcal{F}[l,U]}.$$

One of the important facts about the theory of species is that we can perform operations on species: sum, product, composition, derivation, and integration.

Définition 1.1.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be two species.*

- *The sum species is defined by $(\mathcal{F} + \mathcal{G})[l] = \mathcal{F}[l] \sqcup \mathcal{G}[l]$.*
- *The product species is defined by*

$$(\mathcal{F} \times \mathcal{G})[l] = \sum_{(l_1, l_2)} \mathcal{F}[l_1] \times \mathcal{G}[l_2],$$

where the sum runs over all decompositions of l into an ordered pair of disjoint subsets (l_1, l_2) with $l_1 \sqcup l_2 = l$.

- *If $\mathcal{G}[\emptyset] = \emptyset$, the composition of $\mathcal{F}$ by $\mathcal{G}$ is defined by:*

$$(\mathcal{F} \circ \mathcal{G})[l] = \sum_{\pi \in \mathcal{P}ar[l]} \mathcal{F}[\pi] \times \prod_{b \in \pi} \mathcal{G}[b],$$

where the sum runs over all set partitions π of l.

- *If $\mathcal{F}$ is an $\mathbb{L}$-species, we define the derived and integral species*[3] *by*

$$\begin{aligned}\mathcal{F}'[[n]] =& \frac{d}{dT}\mathcal{F} = \mathcal{F}[0 \oplus [n]];\\ (\int \mathcal{F})[\emptyset] =& \emptyset;\\ (\int \mathcal{F})[l] =& \mathcal{F}[l \setminus \{\min(l)\}].\end{aligned}$$

Now let us state some immediate properties of these definitions:

· $(\mathcal{F}+\mathcal{G})(t) = \mathcal{F}(t)+\mathcal{G}(t)$;

· $(\mathcal{F}\mathcal{G})(t) = \mathcal{F}(t)\mathcal{G}(t)$;

· $(\mathcal{F}\circ\mathcal{G})(t) = \mathcal{F}(\mathcal{G}(t))$;

· $\mathcal{F}'(t) = \dfrac{d}{dt}(\mathcal{F}(t))$;

· $(\mathcal{F}+\mathcal{G})' = \mathcal{F}'+\mathcal{G}'$;

· $(\mathcal{F}\mathcal{G})' = \mathcal{F}'\mathcal{G}+\mathcal{F}\mathcal{G}'$;

· $(\mathcal{F}\circ\mathcal{G})' = \mathcal{F}'(\mathcal{G})\mathcal{G}'$;

· $\int \mathcal{F}' = \mathcal{F}^+,\ \int(\mathcal{F})' = \mathcal{F},\ (\int \mathcal{F})(t) = \int_0^t \mathcal{F}(x)\,dx,$
$\int(\mathcal{F}+\mathcal{G}) = \int\mathcal{F}+\int\mathcal{G}$ and $\int \mathcal{F}'(\mathcal{G})\mathcal{G}' = (\mathcal{F}\circ\mathcal{G})^{+}$[4].

A more complete and detailed treatment of these theories can be found in Bergeron [16] or Randrianirina [17].

1.1.3 Solving combinatorial differential equations

This method is essentially due to P. Leroux and G.X. Viennot (see 18). In Randrianirina [17] or Bergeron [16], any formal series in $\mathbb{Q}[[x,t]]$ of the form

$$F(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!}, \quad \text{with } a_n \in\ \mathbb{A} = \mathbb{Z}[[x]],$$

is a generating series of a (non-unique) weighted $\mathbb{L}$-species.

Thus, to every analytic equation $y' = F(y)$, one can associate a combinatorial equation:

$$Y' = \mathcal{F}(Y),$$

where $\mathcal{F} = \mathcal{F}(T)$ is a species whose generating series is $\mathcal{F}(t) = F(t)$.

[3] One can still define a derivative in the category of $\mathbb{B}$-species.
[4] $\mathcal{F}^+$ is a species identical to $\mathcal{F}$ except that $\mathcal{F}^+[\emptyset] = \emptyset$.

In full generality, given any $\mathbb{L}$-species $\mathcal{F}$, we may write the combinatorial differential equation:

$$Y' = \mathcal{F}(Y); \quad Y(0) = Z \tag{1.1}$$

where Z is a sort of point representing the initial condition.

Intuitively, to solve (1.1), we seek a species $Y = \mathcal{G} = \mathcal{G}(T, Z)$ satisfying

$$\frac{\partial}{\partial T}\mathcal{G} = \mathcal{F}(\mathcal{G}) \quad \text{and} \quad \mathcal{G}(0, Z) = Z.$$

In other words, equation (1.1) can also be written as:

$$\frac{\partial}{\partial T}Y = \mathcal{F}(Y) \quad \text{with} \quad Y(0, Z) = Z \tag{1.2}$$

Définition 1.1.6 ([17])**.** A solution of equation (1.1) is a pair $(\mathcal{G}, \phi)$ where:

1. $\mathcal{G} = \mathcal{G}(T, Z)$ is an $\mathbb{L}$-species if $\mathcal{F}$ is an $\mathbb{L}$-species, or a mixed species if $\mathcal{F}$ is a $\mathbb{B}$-species.

2. $\phi : \dfrac{\partial}{\partial T}\mathcal{G} \to \mathcal{F}(\mathcal{G})$ is an isomorphism of $\mathbb{L}$-species or of mixed species.

Remarque 1.1.2. *To equation (1.1), we associate the analytic differential equation*

$$\frac{\partial y}{\partial t} = \mathcal{F}(y) \ ; \ y(0, z) = z.$$

The equation (1.1) can be written in integral form as:

$$Y(T, Z) = Z + \int_0^T \mathcal{F}(Y(X, Z))\, dX \tag{1.3}$$

The integral equation (1.3) is interpreted by Figure 1.1, which represents an iterative process allowing us to construct the combinatorial solution of equation (1.1). The general solution is an increasing $\mathcal{F}$-enriched tree.

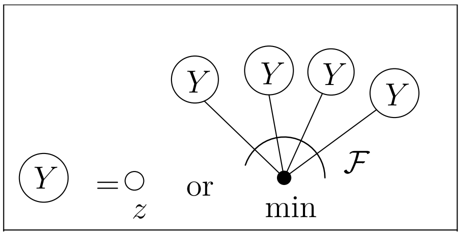


Figure 1.1: Integral equation

Given k-sorted species $(\mathcal{F}_i)_{1 \le i \le k}$ which are either all $\mathbb{B}$-species or all $\mathbb{L}$-species, and sorts of points Z_i representing the initial conditions, consider the system of combinatorial differential equations:

$$Y_i' = \mathcal{F}_i(Y_1, Y_2, ..., Y_k), \quad Y_i(0) = Z_i, \quad 1 \le i \le k. \tag{1.4}$$

Définition 1.1.7. *A solution of system (1.4) is a pair $(\vec{A}, \phi)$, where $\vec{A} = (A_1, A_2, .., A_k)$ is a k-tuple of mixed species of type $\mathbb{L} \times \mathbb{B}^k \to \mathbb{E}_A$ if all the $\mathcal{F}_i$ are $\mathbb{B}$-species, or $(k+1)$-sorted $\mathbb{L}$-species if all the $\mathcal{F}_i$ are $\mathbb{L}$-species, and $\phi = (\phi_1, \phi_2, ..., \phi_k)$ is a k-tuple of isomorphisms*

$$\phi_i : \frac{\partial A_i}{\partial T} \to \mathcal{F}_i(\vec{A}).$$

Let $\vec{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k)$, $\vec{Y} = (Y_1, Y_2, ..., Y_k)$ and $\vec{Z} = (Z_1, Z_2, ..., Z_k)$. System (1.4) can be written in integral form as

$$Y_i(T, Z) = \int_0^T \mathcal{F}_i(\vec{Y}(X, \vec{Z}))\, dX, \quad Y_i(0) = Z_i. \tag{1.5}$$

This system of integral equations (1.5) is interpreted by Figure 1.2. To construct a Y_i-structure, it suffices to iterate the process.

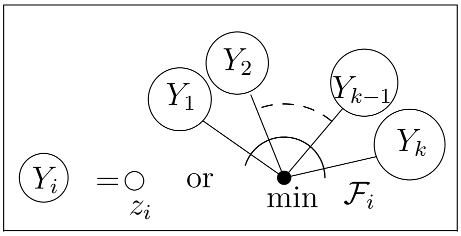


Figure 1.2: System of integral equations

1.2 Viewing in $\mathbb{N} \times \mathbb{N}$

We now introduce the objects that will be studied later in Chapter 2. The grid $\mathbb{N} \times \mathbb{N}$ can be visualized as in Figure 1.3.

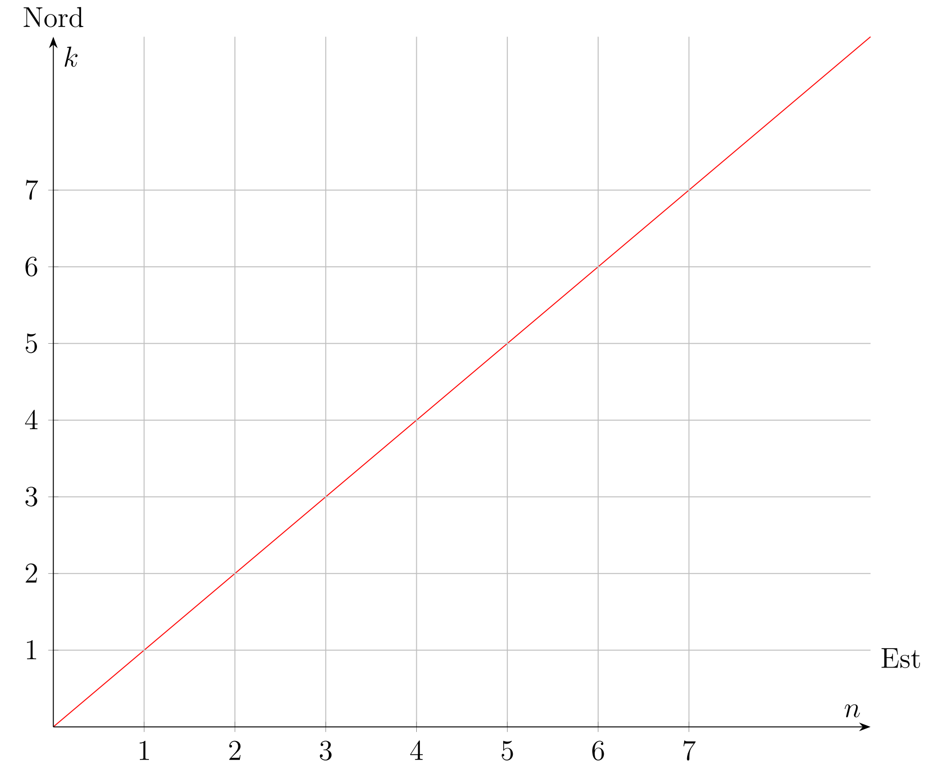


Figure 1.3: Grid $\mathbb{N} \times \mathbb{N}$

Let $\mathcal{R}_{n,k}$ denote the set of paths in the grid $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to (n,k), formed only by East and North–East steps, and let $R_n = \bigcup_{k\in\mathbb{N}} \mathcal{R}_{n,k}$ be the set of all such paths of length n. Similarly, $R^{(k)} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_{n,k}$ be the set of all such paths of height k. Figure 1.4 shows an example of a path in $\mathcal{R}_{n,k}$ with $n = 8$ and $k = 3$.

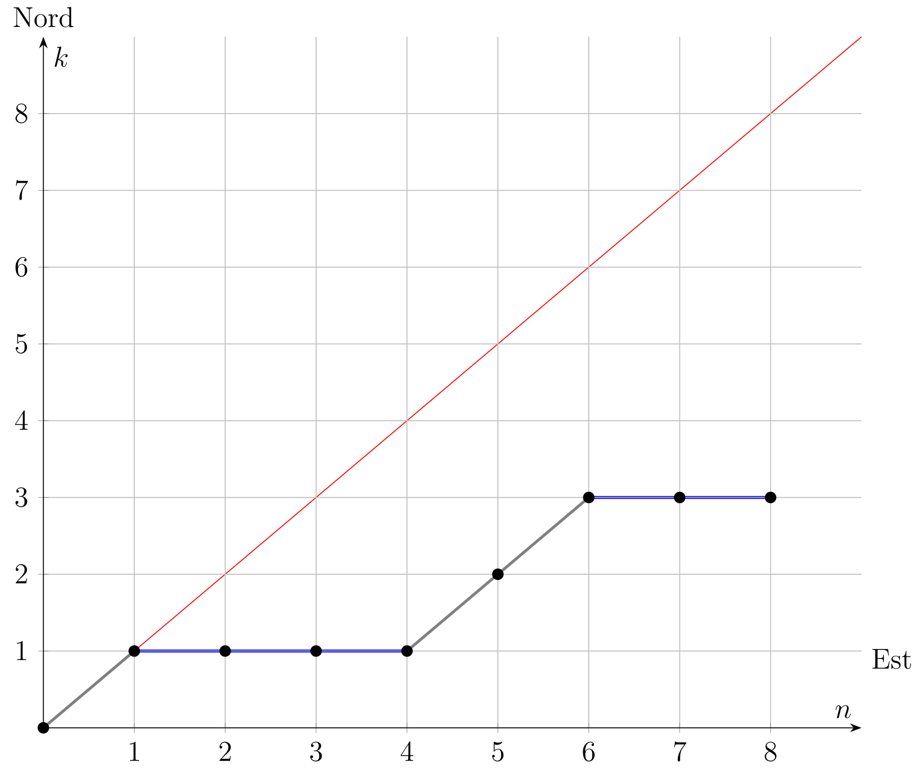


Figure 1.4: Path NE–E–E–E–NE–NE–E–E.

Since we only use East and North–East steps, for $n < k$, the set $\mathcal{R}_{n,k}$ is empty. Before weighting these sets, let us first note that $R^{(k)}$ is infinite and the respective cardinalities of $\mathcal{R}_{n,k}$ and R_n are

$$r_{n,k} := |\mathcal{R}_{n,k}| = \binom{n}{k} \quad \text{and} \quad r_n := |R_n| = 2^n.$$

Now consider two sequences of real numbers $(a_{n,k})_{n,k\in\mathbb{N}}$ and $(b_{n,k})_{n,k\in\mathbb{N}}$. To study the recurrence relation (1), we define a weight p on the set $\mathcal{R}_{n,k}$ and on R_n as follows:

(i) each East step from $(i-1, j)$ to (i, j) has weight $a_{i,j}$,

(ii) each North–East step from $(i-1, j-1)$ to (i, j) has weight $b_{i,j}$,

(iii) for every $c \in \mathcal{R}_{n,k}$ with $c = p_1 p_2 \cdots p_n$, we set

$$p(c) = \prod p(p_i),$$

where the weights $p(p_i)$ are defined as in (i) and (ii).

Sometimes we need other weights: for each $c \in \mathcal{R}_{n,k}$, we define $q(c) = p(c)x^k$ and $u(c) = p(c)y^n$ (that is, x records the height of the path c (the number of North–East steps) and y the number of steps).

Lemme 1.2.1. *In this setting, we have:*

$$T(n,k) = [x^k]|R_n|_q = [y^n]|R^{(k)}|_u = |\mathcal{R}_{n,k}|_p. \tag{1.6}$$

We will explain this statement and, more importantly, exploit it to obtain some characterizations of sequences $(T(n,k))_{n,k\in\mathbb{N}}$ satisfying (1), in chapter 2.

Proof. First, let us prove the equality $T(n,k) = |\mathcal{R}_{n,k}|_p$ by showing that $\left(|\mathcal{R}_{n,k}|_p\right)$ satisfies (1). For $n = 0$, the equality is trivial. Suppose it holds up to $n-1$ (for $n \geq 1$). To reach (n,k), there are two possible last steps: from $(n-1,k)$ or from $(n-1,k-1)$. Thus

$$|\mathcal{R}_{n,k}|_p = \sum_{\sigma\in\mathcal{R}_{n,k}} p(\sigma) = \sum_{\sigma'\in\mathcal{R}_{n-1,k}} a_{n,k}p(\sigma') + \sum_{\sigma'\in\mathcal{R}_{n-1,k-1}} b_{n,k}p(\sigma') = a_{n,k}|\mathcal{R}_{n-1,k}|_p + b_{n,k}|\mathcal{R}_{n-1,k-1}|_p.$$

This gives the desired recurrence. For the remaining equalities, we simply use the definition of total weight:

$$|R_n|_q = \sum_{\sigma\in R_n} q(\sigma) = \sum_k \sum_{\sigma\in\mathcal{R}_{n,k}} q(\sigma) = \sum_k \sum_{\sigma\in\mathcal{R}_{n,k}} p(\sigma)x^k = \sum_k |\mathcal{R}_{n,k}|_p x^k;$$

$$|R^{(k)}|_u = \sum_{\sigma\in R^{(k)}} u(\sigma) = \sum_n \sum_{\sigma\in\mathcal{R}_{n,k}} u(\sigma) = \sum_n \sum_{\sigma\in\mathcal{R}_{n,k}} p(\sigma)y^n = \sum_n |\mathcal{R}_{n,k}|_p y^n.$$

Thus we obtain the equation (1.6). $\square$

1.3 Study of particular cases according to the nature of the sequences $(a_{i,j})_{i,j\in\mathbb{N}}$ and $(b_{i,j})_{i,j\in\mathbb{N}}$

Our aim in this section is simply to give a first taste of the method and to highlight the cases where explicit formulas can be obtained in a straightforward way. We deliberately start with the binomial case where $a_{i,j} = b_{i,j} = 1$. The relations can be found analytically by manipulating certain generating series. We will also try to provide combinatorial methods. In what follows, we assume that $T(0,0) = 0$ and $T(n,k) = 0$ for $n < \max(0,k)$.

1.3.1 Case 0: $a_{i,j} = b_{i,j} = 1$

We can obtain an explicit formula for these numbers in several ways and from different viewpoints, since the well-known recurrence

$$T(n,k) = T(n-1,k) + T(n-1,k-1)$$

is the binomial relation. In our approach, we have:

$$\forall\, c \in \mathcal{R}_{n,k}, \quad q(c) = x^k \quad \text{i.e.} \quad \forall c \in R_n,\ q(c) = x^{h(c)}, \tag{1.7}$$

where $h(c)$ denotes the height of the path c (the number of North–East steps), and

$$|R_n|_q = \sum_{c\in R_n} x^{h(c)} = \sum_{0\leq k\leq n} r_{n,k}x^k. \tag{1.8}$$

Therefore, the sequence $(T_0(n,k))_{n,k\in\mathbb{N}}$ is nothing but the binomial coefficient $\binom{n}{k}$, that is, $T_0(n,k) = r_{n,k} = \binom{n}{k}$.

We also note that, in Hao [8], one defines

$$T_{0,n}(x) = \sum_{0\leq k\leq n} T_0(n,k)x^k.$$

In this case, $T_{0,n}(x) = (1+x)^n$.

1.3.2 Case 1: $a_{i,j} = a$ and $b_{i,j} = b$ constants

First, observe that

$$\forall c \in \mathcal{R}_{n,k}, \quad p(c) = a^{n-k}b^k \quad \text{ and } \quad q(c) = a^{n-k}b^k x^k.$$

Thus

$$|R_n|_p = \sum_{c \in R_n} p(c) = \sum_{k \in \mathbb{N}} \sum_{c \in \mathcal{R}_{n,k}} a^{n-k}b^k = \sum_{0 \le k \le n} r_{n,k} a^{n-k} b^k,$$

and

$$|R_n|_q = \sum_{c \in R_n} q(c) = \sum_{0 \le k \le n} r_{n,k} a^{n-k} x^k = \sum_{0 \le k \le n} \binom{n}{k} a^k (bx)^{n-k} = (a+bx)^n. \tag{1.9}$$

From equations (1.6) and (1.9), if $a_{i,j} = a$ and $b_{i,j} = b$, then

$$T(n,k) = \binom{n}{k} a^{n-k} b^k. \tag{1.10}$$

Remarque 1.3.1. If a, b are positive integers, we can interpret the weighting as a type or coloring. That is, we have a possible colors for each East step and b possible colors for each North–East step. In this case, $T(n,k)$ can be interpreted as the number of paths from $(0,0)$ to (n,k) using exactly $n-k$ East steps and k North–East steps, where each East step has a possible colors and each North–East step has b possible colors. Since there are $a^{n-k}b^k$ possible colorings of a path and $|\mathcal{R}_{n,k}| = r_{n,k}$, we recover (1.10).

1.3.3 Case 2: $a_{i,j} = ai$ and $b_{i,j} = bi$, with a and b constants

This case differs from the previous one only in that the weight of the i-th step is multiplied by i, so each path in $\mathcal{R}_{n,k}$ has its weight multiplied by $1 \times 2 \times \cdots \times n = n!$ compared to its weight in the previous case. The same holds for the total weight. Hence, if $a_{i,j} = ai$ and $b_{i,j} = bi$, then

$$T(n,k) = n! \binom{n}{k} a^{n-k} b^k. \tag{1.11}$$

1.3.4 Case 3: $a_{i,j} = aj$ and $b_{i,j} = bj$, with a and b constants

Let

$$D_k(x) = \sum_{n \ge 0} T(n,k) x^n. \tag{1.12}$$

By manipulating this series using the recurrence relation, we obtain, for $k \ge 1$:

$$\begin{aligned} D_k(x) =& T(0,k) + \sum_{n \ge 1} T(n,k) x^n \\ =& \sum_{n \ge 1} akT(n-1,k) x^n + \sum_{n \ge 1} bkT(n-1,k) x^n \\ =& akx \sum_{n \ge 0} T(n,k) x^n + bkx \sum_{n \ge 0} T(n,k-1) x^n \\ =& akx D_k(x) + bkx D_{k-1}(x). \end{aligned}$$

Thus

$$D_k(x) = \frac{bkx}{1-akx} D_{k-1}(x). \tag{1.13}$$

We have $D_0(x) = \sum_{n\in\mathbb{N}} T(n,0)x^n = 1$ since $T(n,0) = 0 \times T(n-1,0)$ for all $n \geq 0$. Using equation (1.13) and this value of D_0, we obtain

$$D_k(x) = k!b^k a^{-k}\left((ax)^k \prod_{i=1}^{k} \frac{1}{1-aix}\right). \tag{1.14}$$

Using the generating function of the Stirling numbers $S(n,k)$ (see Comtet [19]), we may also write

$$D_k(x) = k!b^k \sum_{n\geq k} S(n,k)a^{n-k}x^n. \tag{1.15}$$

Thus, combining equations (1.12) and (1.15), if $a_{n,k} = ak$ and $b_{n,k} = bk$, we get

$$T(n,k) = k!b^k a^{n-k} S(n,k). \tag{1.16}$$

Combinatorial proof

In this case, each path:

(i) has exactly k North–East steps

$$((\sigma(1),0) \to (\sigma(1)+1,1)),\ ((\sigma(2),1) \to (\sigma(2)+1,2)),\ \ldots,\ ((\sigma(k),k-1) \to (\sigma(k)+1,k)),$$

for a strictly increasing map $\sigma : [k] \to \{0,1,\cdots,n-1\}$, and these contribute a weight

$$b \times 2b \times 3b \times \cdots \times kb = k!b^k;$$

(ii) has exactly $n-k$ East steps, each lying on some level $j \in \{0,1,\ldots,k\}$, that is, with weight $0^{c_0}(a)^{c_1}(2a)^{c_2}\cdots(ka)^{c_k}$ where $c_0+c_1+c_2+\cdots+c_k = (n-k)$ (the number of East steps). Hence the total weight is

$$a^{n-k} \sum_{c_0+c_1+\cdots+c_k=n-k} 0^{c_0}1^{c_1}\cdots k^{c_k} = a^{n-k}S(n,k),$$

which is also a classical formula found in Comtet [19];

(iii) the increasing map σ in (i) is uniquely determined by the (c_i) in (ii).

Using equation (1.6), we obtain

$$T(n,k) = |\mathcal{R}_{n,k}|_p = \sum_{c_1+c_2+\cdots+c_{k+1}=n-k} k!b^k a^{n-k}1^{c_1}2^{c_2}\cdots k^{c_k} = k!b^k a^{n-k}S(n,k),$$

which completes the combinatorial proof of equation (1.16).

Remarque 1.3.2. With a similar reasoning, we can obtain for $b_{n,k} = b_1k+b_0$ and $a_{n,k} = ak$:

$$T(n,k) = (b_0+b_1|b_1)^{(\overline{k})}a^{n-k}S(n,k),$$

with $(b_0+b_1|b_1)^{(\overline{k})} = \prod_{i=0}^{k-1}(b_0+b_1+ib_1)$.

These approaches can lead to various results, but we will return later to the more general case in Chapter 2.

1.3.5 General case

Théorème 1.3.1 ([4]). *The general term $T(n,k)$ satisfying* (1) *is given by the formula*

$$\sum_{\substack{\{\alpha_1,\ldots,\alpha_{n-k}\}\sqcup\{\beta_1,\ldots,\beta_k\}=[n]\\ \alpha_i<\alpha_{i+1},\ \beta_j<\beta_{j+1}}} \left(\prod_{i=1}^{n-k} a_{\alpha_i,\,\alpha_i-i}\right)\left(\prod_{j=1}^{k} b_{\beta_j,\,j}\right), \tag{1.17}$$

where the sum runs over all partitions of $[n]$ into two subsets $\{\alpha_1,\ldots,\alpha_{n-k}\}$ and $\{\beta_1,\ldots,\beta_k\}$, with again $\alpha_i<\alpha_{i+1}$ and $\beta_j<\beta_{j+1}$.

The approach in Théorêt [4] is rather analytic, whereas in the present thesis we aim to work with approaches based on grammars, weighted paths, and Riordan matrices.

1.4 Motivation

Although we already have the explicit formula (1.17), many authors continue to study these objects to discover new relations. Even without discussing complexity in detail, it is clear that expressing $\binom{n}{k}$ or $S(n,k)$ using only (1.17) leads to formulas that are quite cumbersome to evaluate, even for computers. This also matters for obtaining good approximations when needed, and more generally for understanding the sequence $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$, such as its asymptotic behavior.

Moreover, these numbers appear in many areas of mathematics and applications. The Stirling numbers first appeared in connection with the resolution of certain differential equations (see Stirling [20]). As discussed by Salas [21], solutions of recurrences of the form (1) are related to certain differential equations. These numbers are also used to evaluate certain integrals, as in Rzadkowski [22] and Cannon [23]. Bell numbers are connected to the Riemann zeta function in number theory: one can express the zeta function in terms of Stirling numbers (see also Mező [24]).

In more applied contexts, some probability distributions are treated with Stirling numbers and Eulerian numbers (see Harris [25], Adell [27]). Wilf [5] also mentions that such sequences arise in problems of bioinformatics, in the context of DNA sequence alignment.

1.5 Approach using grammars

Let $X=\{x_1,x_2,\cdots\}$ be a set and $C[X]$ the commutative algebra of polynomials in the letters x_i. A grammar G on X is a map from X to $C[X]$. In this case, X is the alphabet. The basic notion of a context-free grammar can be found in Chen [28] or Dumont [11].
We restrict ourselves to the case where the alphabet X is finite. For each grammar G, we associate a differential operator

$$\mathcal{D}=\sum_{x\in X} G(x)\frac{\partial}{\partial x}.$$

Let us rewrite the equation (1):

$$T(n,k)=(a_2n+a_1k+a_0)T(n-1,k)+(b_2n+b_1k+b_0)T(n-1,k-1).$$

We try to apply or refine the approach of Zhou [9] and Hao [8]. Define the grammars

$$\begin{aligned} G_1 = \{&x \to (b_1+b_2)x^2 + (a_1+a_2)xy; \\ &y \to b_2xy + a_2y^2; \\ &z \to (b_1+b_2+b_0)zx + (a_0+a_2)zy\} \text{ and} \end{aligned} \tag{1.18}$$

$$\begin{aligned} G_2 = \{&u \to u^{b_1+b_2+1}v^{a_1+a_2}; \\ &v \to u^{b_2}v^{a_2+1}\} \end{aligned} \tag{1.19}$$

and let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the respective differential operators.

Proposition 1.5.1 ([8]). *We have:*

$$\mathcal{D}_2^n(u^{b_0+b_1+b_2}v^{a_0+a_2}) = \sum_{k=0}^{n} T(n,k)u^{b_2n+b_1k+b_0+b_1+b_2}v^{a_2n+a_1k+a_0+a_2}. \tag{1.20}$$

Proposition 1.5.2. *We also have:*

$$\mathcal{D}_1^n(z) = z\sum_{k=0}^{n} T(n,k)x^ky^{n-k}. \tag{1.21}$$

Proof. For $n=0$, this is clear. Now suppose the relation holds up to $n-1$ (for $n \geq 1$), that is, $\mathcal{D}_1^{n-1}(z) = z\sum_{k=0}^{n-1}\mathrm{T}(n-1,k)x^ky^{n-1-k}$, and show that equality (1.21) follows. By Leibniz's rule, we have:

$$\begin{aligned} \mathcal{D}_1^n(z) =& \mathcal{D}\Big(\mathcal{D}_1^{n-1}(z)\Big) \\ =& \mathcal{D}\left(\sum_{k=0}^{n-1}\mathrm{T}(n-1,k)zx^ky^{n-1-k}\right) \\ =& \sum_{k=0}^{n-1}\mathrm{T}(n-1,k)\Big(\mathcal{D}(z)x^ky^{n-1-k} + z\mathcal{D}(x^k)y^{n-1-k} + zx^k\mathcal{D}(y^{n-1-k})\Big). \\ =& \sum_{k=0}^{n-1}\mathrm{T}(n-1,k)\Big(\Big((b_1+b_2+b_0)zx + (a_0+a_2)zy\Big)x^ky^{n-1-k} \\ &\quad + zk\Big((b_1+b_2)x^2 + (a_1+a_2)xy\Big)x^{k-1}y^{n-1-k} \\ &\quad + zx^k(n-1-k)\Big(b_2xy + a_2y^2\Big)y^{n-2-k}\Big) \\ =& \sum_{k\geq 0}\Big(b_2n + b_1(k+1) + b_0\Big)\mathrm{T}(n-1,k)x^{k+1}y^{n-1-k} \\ &\quad + \sum_{k\geq 0}(a_2n + a_1k + a_0)\mathrm{T}(n-1,k)x^ky^{n-k}. \end{aligned}$$

Thus

$$\mathcal{D}_1^n(z) = z\sum_{k\geq 0}\Big((a_2n+a_1k+a_0)T(n-1,k) + (b_2n+b_1k+b_0)T(n-1,k-1)\Big)x^ky^{n-k},$$

which implies the equation (1.21). $\square$

For $x \in C[X]$, we associate the exponential generating function

$$Gen(x,t) := \sum_{n \in \mathbb{N}} \mathcal{D}^n(x) \frac{t^n}{n!}.$$

It is easy to check that $\mathcal{D}$ is a derivation in $C[X]$, that is, a linear map satisfying the Leibniz rule[5]. Thus, for all $x, y \in C[X]$:

$$Gen(x+y,t) = Gen(x,t) + Gen(y,t) \quad \text{and} \quad Gen(xy,t) = Gen(x,t)Gen(y,t). \tag{1.22}$$

It is also clear that

$$\frac{d}{dt} Gen(x,t) = \mathcal{D}\Big(Gen(x,t)\Big).$$

In this way, Dumont [11], stated as a proposition in Randrianirina [10], proves:

Proposition 1.5.3. *Let $\vec{y}(t) = (y_i(t))_{i=1,\dots,l}$ be the solution of the analytic differential system*

$$y_i' = G_i(\vec{y}(t)), \qquad y_i(0) = x_i, \quad i = 1, \dots, l.$$

Then for each i, $\mathrm{Gen}(x_i, t) = y_i(t)$.

The version formulated in the language of species is given in Randrianirina [17] as the following.

Théorème 1.5.1. *Given asymmetric species $(G_i)_{i=1,\dots,l}$, the following data are equivalent:*

1. *the data of the combinatorial differential system*

$$Y_i' = G_i(\vec{Y}); \quad Y_i(0) = X_i, \quad i = 1, \dots, l;$$

2. *the data of the combinatorial differential operator*

$$\mathcal{D} = \sum_{i=1}^{l} G_i(\vec{X}) \frac{\partial}{\partial X_i};$$

3. *the data of William Chen's grammar*

$$G : x_i \to G_i(\vec{x}); \quad i = 1, \dots, l;$$

4. *the data of the system of analytic differential equations*

$$y_i' = G_i(\vec{y}(t)); \quad y_i(0) = x_i, \quad i = 1, \dots, l.$$

Recall that a species $\mathcal{G}$ is asymmetric if, for all $n \in \mathbb{N}$ and for all $s, t \in \mathcal{G}([n])$, the existence of a bijection $\sigma : [n] \to [n]$ with $\sigma(s) = t$ implies $s = t$.

[5]The Leibniz rule means that for $f, g \in C[X]$, we have $\mathcal{D}(fg) = \mathcal{D}(f)\, g + f\, \mathcal{D}(g)$.

Exemple 1.5.1 (r-Whitney–Eulerian numbers $A_{m,r}(n,k)$)**.** *In Foata [29], ${}^rA(n,k)$ is defined as the number of permutations $\sigma \in S_n$ having k r-excedances, where $j \in [n]$ is an r-excedance of σ if $j + r \leq \sigma(j)$. We use here the notation $A_r(n,k)$. The author proves that these numbers satisfy the recurrence:*

$$A_r(n,k) = (k + r)A_r(n-1,k) + (n - k + 1 - r)A_r(n-1,k-1). \tag{1.23}$$

Riordan [30] (see also Maier [31]) discusses combinatorial interpretations of numbers satisfying the recurrence (1.23) *in terms of statistics of r-descents. This is a generalization of the Eulerian numbers $A_r(n,k) = \left\langle {n \atop k} \right\rangle$. It is also the number of permutations $\sigma \in S_n$ having $n-r-k$ indices $i \in [n-1]$ with $\sigma(i) < \sigma(i+1)$ and $\sigma(i) < n-r$. To generalize, we first ignore the fact that this definition requires $r < n$ and allow r to be arbitrary. Second, we imagine that each $i \in [n]$ can have m different types. These ideas lead to the definition of the r-Whitney–Eulerian numbers $A_{m,r}(n,k)$. A combinatorial interpretation is given in Thamrongpairoj [32]. These numbers are defined by the recurrence ($A_{m,r}(n,k) = 0$ if $n < \max(0,k)$):*

$$A_{m,r}(0,0) = 1 \quad \textit{and} \quad A_{m,r}(n,k) = (mk+r)A_{m,r}(n-1,k) + \Big(mn - mk + m - r\Big)A_{m,r}(n-1,k-1).$$

In this case, the differential systems associated with the grammars G_1 and G_2 in (1.18) *and* (1.19) *are respectively*

$$\begin{cases} X' = & mXY, \quad X(0) = x, \\ Y' = & mXY, \quad Y(0) = y, \\ Z' = & (m-r)ZX + rZY, \quad Z(0) = z, \end{cases} \tag{1.24}$$

$$\begin{cases} U' = & UV^m, \quad U(0) = u, \\ V' = & U^mV, \quad V(0) = v. \end{cases} \tag{1.25}$$

The analytic solutions are respectively

$$\begin{cases} X(t) = & \dfrac{(x-y)x}{x - ye^{(x-y)mt}}, \\ Y(t) = & \dfrac{(x-y)y}{xe^{(y-x)mt} - y}, \\ Z(t) = & \dfrac{(x-y)ze^{(m-r)(x-y)t}}{x - ye^{(x-y)mt}}. \end{cases} \tag{1.26}$$

$$\begin{cases} U(t) = & \left(\dfrac{(u^m - v^m)u^m}{u^m - v^m e^{(u^m - v^m)mt}}\right)^{\frac{1}{m}}, \\ V(t) = & \left(\dfrac{(v^m - u^m)v^m}{v^m - u^m e^{(v^m - u^m)mt}}\right)^{\frac{1}{m}}. \end{cases} \tag{1.27}$$

The reader can easily check that (1.26) *and* (1.27) *satisfy* (1.24) *and* (1.25)*, respectively. We then relate these solutions to*

$$F_1(z,t) = \sum_{n \in \mathbb{N}} \mathcal{D}_1^n(z) \frac{t^n}{n!} = Z(t),$$

$$F_2(u^{m-r}v^r, t) = \sum_{n \in \mathbb{N}} \mathcal{D}_2^n(u^{m-r}v^r) \frac{t^n}{n!} = \Big(U(t)\Big)^m \times \left(\frac{V(t)}{U(t)}\right)^r.$$

These facts follow from (1.22) *and Proposition 1.5.3. Applying Propositions 1.5.1 and 1.5.2, we obtain:*

$$\sum_{n\in\mathbb{N}}\left(\sum_{k=0}^{n}A_{m,r}(n,k)x^k y^{n-k}\right)\frac{t^n}{n!}=\frac{(y-x)e^{(y-x)rt}}{y-xe^{(y-x)mt}}; \tag{1.28}$$

$$\sum_{n\in\mathbb{N}}\left(\sum_{k=0}^{n}A_{m,r}(n,k)u^{mn-mk}v^{mk}\right)\frac{t^n}{n!}=\frac{(u^m-v^m)e^{(u^m-v^m)rt}}{u^m-v^m e^{(u^m-v^m)mt}}. \tag{1.29}$$

These equations yield (they are equivalent):

$$\sum_{k=0}^{n}A_{m,r}(n,k)=m^n n!; \tag{1.30}$$

$$\sum_{k=0}^{n}(-1)^k A_{m,r}(n,k)=2^n\sum_{k}\binom{n}{k}m^k E_k(0)r^{n-k}; \tag{1.31}$$

$$\sum_{k\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}}A_{m,r}(n,k)\frac{t^n}{n!}\right)x^k=\frac{(1-x)e^{r(1-x)t}}{1-xe^{(1-x)mt}}. \tag{1.32}$$

This method allows us to obtain the explicit formula for $A_{m,r}(n,k)$. Indeed,

$$\begin{aligned}\frac{(1-x)e^{r(1-x)t}}{1-xe^{(1-x)mt}} &=(1-x)\sum_{j\geq 0}x^j e^{(1-x)(mj+r)t}\\ &=(1-x)\sum_{j\geq 0}x^j\sum_{n\geq 0}\frac{\big((1-x)(mj+r)t\big)^n}{n!}\\ &=\sum_{n\geq 0}\left((1-x)^{n+1}\sum_{j\geq 0}x^j(mj+r)^n\right)\frac{t^n}{n!}\\ &=\sum_{n\geq 0}\left(\sum_{l\geq 0}\binom{n+1}{l}x^l\sum_{j\geq 0}(mj+r)^n x^j\right)\frac{t^n}{n!}\\ &=\sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}}\left(\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}\big(m(k-j)+r\big)^n\right)\frac{t^n}{n!}x^k. \end{aligned}\tag{1.33}$$

Equations (1.32) and (1.33) imply

$$A_{m,r}(n,k)=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}\big(m(k-j)+r\big)^n. \tag{1.34}$$

For $m\geq r$, the grammar G_2 can give us ideas about a structure that interprets the numbers $\big(A_{m,r}(n,k)\big)_{n,k\in\mathbb{N}}$. For example, take $m=3$ and $r=2$. The grammar is then

$$\begin{aligned}G_2=\{u&\to uv^3,\\ v&\to u^3v\},\end{aligned}$$

and the associated differential operator is

$$\mathcal{D}_2=uv^3\frac{\partial}{\partial u}+u^3v\frac{\partial}{\partial v}.$$

$$\mathcal{D}^0 uv^2 = uv^2$$

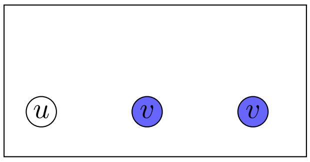


Figure 1.5: $\mathcal{D}^n(u^{m-r}v^r)$-structure for $n = 0$

$$\mathcal{D}uv^2 = uv^5 + 2u^4v^2$$

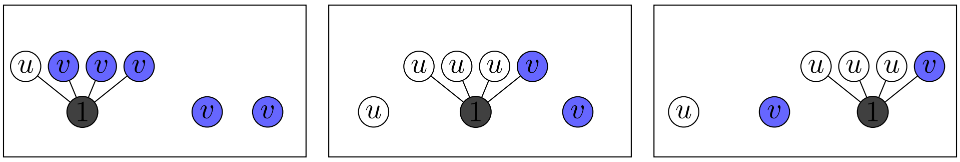


Figure 1.6: $\mathcal{D}^n(u^{m-r}v^r)$-structures for $n = 1$

$$\mathcal{D}^2 uv^2 = uv^8 + 13u^4v^5 + 4u^7v^2$$

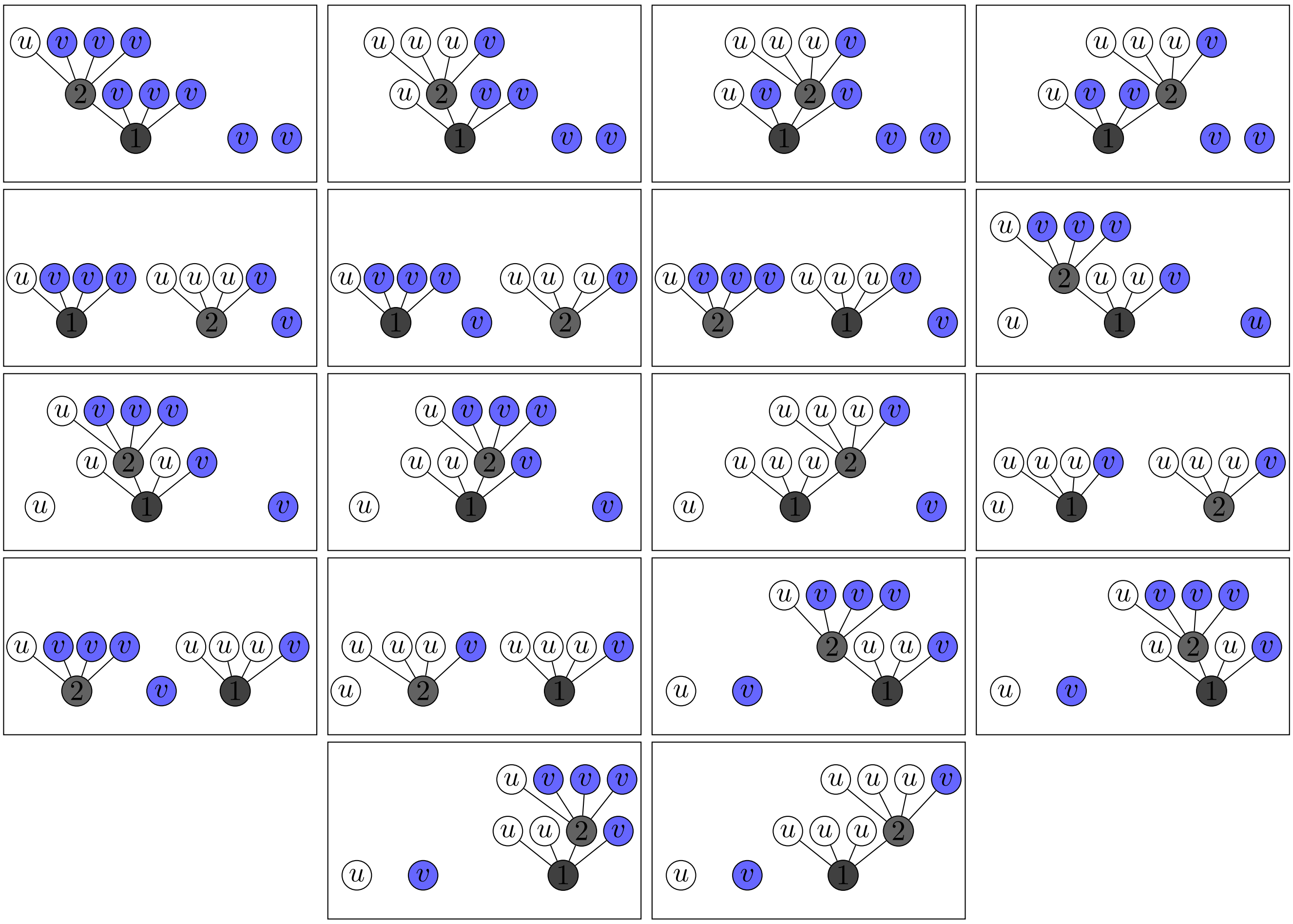


Figure 1.7: $\mathcal{D}^n(u^{m-r}v^r)$-structures for $n=2$

From Proposition 1.20, $A_{m,r}(n,k)$ is the number of $\mathcal{D}^n(u^{m-r}v^r)$-structures having $km+r$ v-leaves (blue in Figures 1.5, 1.6, 1.7).
Furthermore, we note that a $\mathcal{D}^n(u^{m-r}v^r)$-structure has $f_n = m(n+1)$ leaves. If a_n is the number of all $\mathcal{D}^n(u^{m-r}v^r)$-structures, then $a_0 = 1$ and for $n \geq 1$, $a_n = a_{n-1} \times f_{n-1} = nma_{n-1}$. Thus

$$a_n = n!m^n,$$

which confirms relation (1.30).

Exemple 1.5.2. *Consider the sequence $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfying* (1) *with $b_{n,k}=1$ and $a_{n,k} = a_0 + a_1 k$, $a_1 \neq 0$. The differential system associated with the grammar G_2 in* (1.19) *is*

$$\begin{cases} U' = & UV^{a_1}, \quad U(0) = u, \\ V' = & V, \qquad V(0) = v. \end{cases}$$

The analytic solution is

$$\begin{cases} U(t) = & u\exp\left(\frac{v^{a_1}}{a_1}\Big(e^{a_1 t}-1\Big)\right), \\ V(t) = & ve^t. \end{cases} \tag{1.35}$$

This can be checked directly. Recall that the Bell polynomial is defined by

$$B_n(\lambda) = \sum_{k=0}^{n} S(n,k)\lambda^k.$$

A classical identity involving Stirling numbers yields

$$\sum_{n\in\mathbb{N}} B_n(\lambda)\frac{t^n}{n!} = \exp\Big(\lambda(e^t-1)\Big). \tag{1.36}$$

From equations (1.35) and (1.36), we obtain

$$U(t)\Big(V(t)\Big)^{a_0} = uv^{a_0}\sum_{n\in\mathbb{N}}\left(\sum_{k=0}^{n}\binom{n}{k}a_1^k a_0^{n-k}B_k\Big(\frac{v^{a_1}}{a_1}\Big)\right)\frac{t^n}{n!}. \tag{1.37}$$

On the other hand, Proposition 1.5.1 gives

$$\mathcal{D}^n(uv^{a_0}) = \sum_{k=0}^{n} F(n,k)uv^{a_1 k + a_0}. \tag{1.38}$$

Combining these facts with equation (1.22) and Proposition 1.5.3, we obtain

$$\sum_{k\in\mathbb{N}} F(n,k)\alpha^k = \sum_{k\in\mathbb{N}}\binom{n}{k}a_1^k a_0^{n-k}B_k\Big(\frac{\alpha}{a_1}\Big), \tag{1.39}$$

upon substituting $\alpha = v^{a_1}$. Returning to the expression of $B_k(x)$, we then have

$$F(n,k) = \sum_{j=k}^{n} \binom{n}{j} a_0^{n-j} \, a_1^{j-k} \, S(j,k). \tag{1.40}$$

Next, consider $\mathcal{D}^n(uv^{a_0})$ with

$$\mathcal{D} = uv^{a_1}\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

We look at the example of the case $a_0 = 2$ and $a_1 = 2$.

$$\mathcal{D}^0 uv^2 = uv^2$$

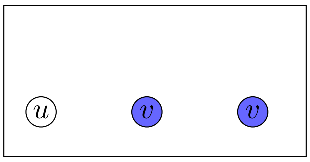


Figure 1.8: $\mathcal{D}^n(uv^{a_0})$-structure for $n = 0$

$$\mathcal{D}uv^2 = uv^4 + 2uv^2$$

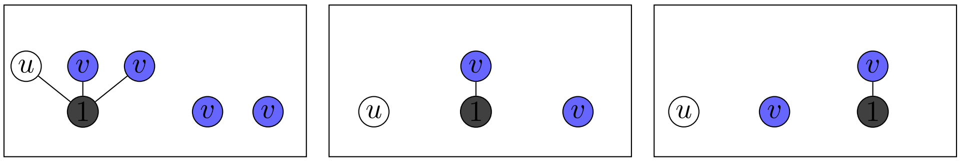


Figure 1.9: $\mathcal{D}^n(uv^{a_0})$-structures for $n = 1$

$$\mathcal{D}^2uv^2 = uv^6 + 4uv^4 + 2uv^4 + 4uv^2$$

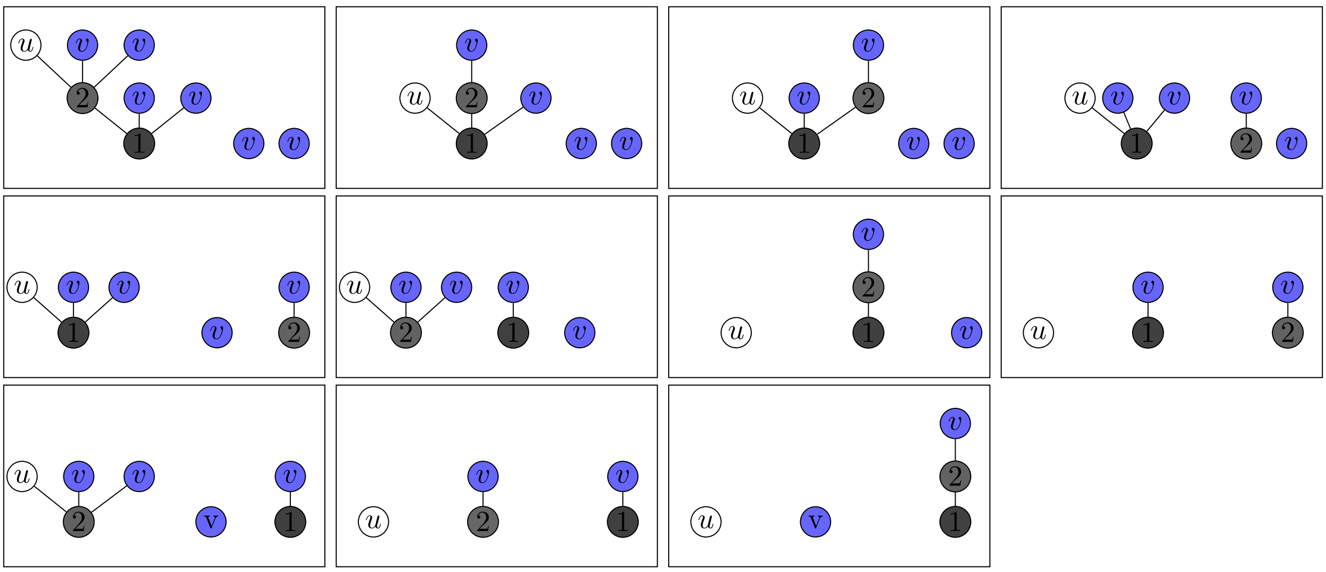


Figure 1.10: $\mathcal{D}^n(uv^{a_0})$-structures for $n = 2$

From equation (1.38), $F(n,k)$ is the number of $\mathcal{D}^n(uv^{a_0})$-structures having a_1k+a_0 v-leaves (blue in Figures 1.8, 1.9, 1.10).

Exemple 1.5.3. *Consider the sequence* $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ *satisfying* (1) *with* $b_{n,k}=1$ *and* $a_{n,k}=a_0+a_2n$, $a_2\neq 0$. *The differential system associated with the grammar* G_2 *in* (1.19) *is*

$$\begin{cases} U' = & UV^{a_2}, \quad U(0)=u, \\ V' = & V^{a_2+1}, \quad V(0)=v. \end{cases}$$

The analytic solution is

$$\begin{cases} U(t) = & \frac{u}{v}(v^{-a_2}-a_2t)^{-\frac{1}{a_2}}, \\ V(t) = & (v^{-a_2}-a_2t)^{-\frac{1}{a_2}}. \end{cases} \tag{1.41}$$

Again, this can be checked by direct computation. Thus we have

$$U(t)\Big(V(t)\Big)^{a_0+a_2} = \frac{u}{v}(v^{-a_2}-a_2t)^{-\alpha} = uv^{a_0+a_2}\sum_{n\geq 0}(\alpha)^{(\underline{n})}a_2^n v^{a_2n}\frac{t^n}{n!},$$

where $\alpha=\frac{1+a_0+a_2}{a_2}$. As before, using Proposition 1.5.1 we get

$$\sum_{n\in\mathbb{N}}\sum_{k=0}^{n}F(n,k)uv^{a_0+a_2+a_2n}\frac{t^n}{n!} = uv^{a_0+a_2}\sum_{n\geq 0}(\alpha)^{(\underline{n})}a_2^n v^{a_2n}\frac{t^n}{n!}.$$

This implies

$$\sum_{k=0}^{n}F(n,k) = (\alpha)^{(\underline{n})}a_2^n = \left(\frac{1+a_0+a_2}{a_2}\right)^{(\underline{n})}a_2^n.$$

Remarque 1.5.1. Briefly, Proposition 1.5.1 states that

$$\mathcal{D}^n(uv^{a_0+a_2}) = \sum_k T(n,k)uv^{a_2n+a_2+a_0},$$

which does not allow us to distinguish the $T(n,k)$ according to the number of v-leaves: all $\mathcal{D}^n(uv^{a_0+a_2})$-structures have the same numbers of leaves, namely one u-leaf and $a_2n+a_2+a_0$ v-leaves.

Therefore, we must introduce additional properties to determine $T(n,k)$. Let us now briefly discuss certain types of grammars: grammars of type (E), introduced in Randrianirina [33].

Définition 1.5.1 ([33])**.** *A grammar of type* (E) *is a grammar* G *on* $\{z\}\cup X$, *with* $z\notin X$, *defined by*

$$\begin{aligned} G = \{ & z \to zg(\vec{x}), \\ & x_i \to h_i(\vec{x}) : \quad x_i\in X\}, \end{aligned}$$

where $g(\vec{x})$ *is a polynomial in* $\mathbb{C}[X]$.

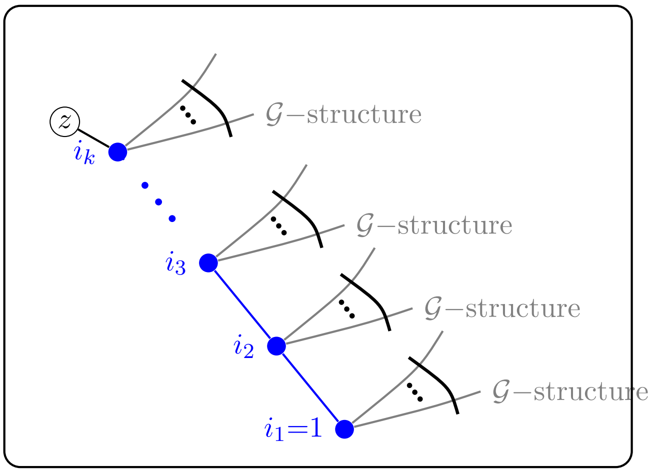


Figure 1.11: $\mathcal{Z}$-structure in function of $\mathcal{G}$-structure.

We associate to this grammar a system of differential equations. Solving it combinatorially, the $\mathcal{Z}$-structure is a species of structures as in Figure 1.11. Thus, $\mathcal{Z}$ is the species of $\mathcal{G}$-structures with

$$\mathcal{G} = \int g\big(\vec{\mathcal{X}}\big).$$

Consider now the blue line, called the backbone (or "spine"). In the terminology of Randrianirina [33], each point on the backbone corresponds to a connected component, and the author associates the species $\mathcal{Z}$ to a sequence $\big(Z(n,k)\big)_{n,k\in\mathbb{N}}$, where $Z(n,k)$ is the number of $\mathcal{Z}$-structures on n points having k connected components.

Now consider a sequence $\big(F(n,k)\big)_{n,k\in\mathbb{N}}$ satisfying (1) with $a_{n,k} = a_2n + a_1k + a_0$, $b_{n,k} = 1$, and the usual initial conditions. The grammar G_2 in (1.19) associated with this sequence is

$$\begin{aligned} G_2 = \{u &\to uv^{a_1+a_2};\\ v &\to v^{a_2+1}\}. \end{aligned}$$

This grammar is of type (E), and we have the following important remark.

Proposition 1.5.4. *Every $\mathcal{D}_2^n(uv^{a_0+a_2})$-structure s decomposes canonically as $s = (s_1, s_2)$, where s_1 is a $\mathcal{U}$-structure and s_2 is a $\mathcal{V}^{a_0+a_2}$-structure. Hence $F(n,k)$ is the number of $\mathcal{D}_2^n(uv^{a_0+a_2})$-structures s such that s_1 has k connected components.*

Proof. First, Proposition 1.5.1 ensures that $F(n,k)$ is the number of $\mathcal{D}_2^n(uv^{a_0+a_2})$-structures having $a_2n+a_1k+a_0+a_2$ v-leaves. Now let $G(n,k)$ denote the number of $\mathcal{D}_2^n(uv^{a_0+a_2})$-structures s whose s_1 has k connected components.

We argue by induction. For $n = 0$, the unique structure has $a_0 + a_2$ v-leaves, so the statement is trivially true. For $n \geq 1$, assume the statement holds up to $n-1$, so that $T(n,k) = G(n,k)$, and prove it for n. Consider a $\mathcal{D}_2^n(uv^{a_0+a_2})$-structure s with k connected components. We focus on the position of the point n. There are two possible cases:

- Case 1: the point n lies on the backbone, say on i_k. This occurs if and only if the last derivation was applied to the unique u-leaf. This last step increases the number of connected components.

- Case 2: the point n is not on the backbone. Hence the last derivation was performed on one of the v-leaves. In this case, the last step does not increase the number of connected components. Using the induction hypothesis and Proposition 1.5.1, there are $a_2(n-1)+a_1k+a_0$ choices for this last step.

This correspondence shows that the sequence $\big(G(n,k)\big)_{n,k\in\mathbb{N}}$ satisfies

$$G(n,k) = G(n-1,k-1) + (a_2 n + a_1 k + a_0)G(n-1,k),$$

which completes the proof. $\square$

Corollaire 1.5.1. *Let $(T(n,k))$ be a sequence satisfying the usual initial conditions and*

$$T(n,k) = T(n-1,k-1) + (a_2 n + a_1 k - a_2)T(n-1,k),$$

and associated with the combinatorial differential system

$$\begin{cases} U' = & UV^{a_1+a_2}, \quad U(0) = u, \\ V' = & V^{a_2+1}, \quad V(0) = v. \end{cases}$$

Then $T(n,k)$ represents the number of $\mathcal{U}$-structures having k connected components.

Théorème 1.5.2. *Let $\big(F(n,k)\big)$ be a sequence satisfying* (1) *with $a_{n,k} = a_2n+a_1k+a_0$, $b_{n,k} = 1$, and the usual initial conditions. If $a_1 \neq 0$ and $a_2 \neq 0$, then*

$$F(n,k) = \frac{1}{a_1^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{r=1}^{n} \Big(a_0 + a_1 j + r a_2\Big). \tag{1.42}$$

Proof. The grammar G_2 in (1.19) associated with this sequence is

$$\begin{aligned} G_2 = \{u &\to u v^{a_1+a_2}, \\ v &\to v^{a_2+1}\}. \end{aligned}$$

We consider the system

$$\begin{cases} U' = U\,V^{\,a_1+a_2}, & U(0) = u, \\ V' = V^{\,a_2+1}, & V(0) = v. \end{cases}$$

By separation of variables, we obtain

$$V(t) = \Big(v^{-a_2} - a_2 t\Big)^{-1/a_2}, \qquad U(t) = u\,\exp\left(\frac{(v^{-a_2} - a_2 t)^{-a_1/a_2} - v^{a_1}}{a_1}\right),$$

valid whenever $v^{-a_2} - a_2 t \neq 0$. Then we have

$$U(t)\,V(t)^{\,a_0+a_2} = u\,v^{\,a_0+a_2}\,(1 - a_2 t\, v^{a_2})^{-\frac{a_0+a_2}{a_2}}\; \exp\left(\frac{v^{a_1}}{a_1}\Big((1 - a_2 t\, v^{a_2})^{-\frac{a_1}{a_2}} - 1\Big)\right).$$

We expand the exponential and the binomial:

$$U(t)\,V(t)^{\,a_0+a_2} = \sum_{k\in\mathbb{N}} \frac{1}{k!}\Big(\frac{v^{a_1}}{a_1}\Big)^k \Big((1-a_2t\,v^{a_2})^{-\frac{a_1}{a_2}}-1\Big)^k$$
$$= \sum_{k\in\mathbb{N}} \frac{1}{k!}\Big(\frac{v^{a_1}}{a_1}\Big)^k \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(1-a_2t\,v^{a_2})^{-\frac{a_0+a_2+a_1j}{a_2}}.$$

For all $n\geq 0$,

$$(1-a_2t\,v^{a_2})^{-\beta} = \sum_{n\in\mathbb{N}}\left(\prod_{r=1}^{n}(\beta+r-1)\right)\frac{(a_2t\,v^{a_2})^n}{n!}.$$

Hence

$$U(t)\,V(t)^{\,a_0+a_2} = \sum_{k\in\mathbb{N}}\frac{1}{a_1^k k!}\sum_{n\in\mathbb{N}}(-1)^{\,k-j}\binom{k}{j}\ \prod_{r=1}^{n}\Big(a_0+a_1j+ra_2\Big)\frac{t^n}{n!}. \tag{1.43}$$

On the other hand, Proposition 1.5.3 implies

$$U(t)\,V(t)^{\,a_0+a_2} = Gen(uv^{\,a_0+a_2},t) = \sum_{n\in\mathbb{N}}\Big(\sum_{k=0}^{n}T(n,k)\ u\ v^{\,a_2n+a_1k+a_2+a_0}\Big)\frac{t^n}{n!}. \tag{1.44}$$

Comparing (1.43) and (1.44), we obtain, for $0\leq k\leq n$,

$$F(n,k) = \frac{1}{a_1^k\,k!}\ \sum_{j=0}^{k}(-1)^{\,k-j}\binom{k}{j}\ \prod_{r=1}^{n}\Big(a_0+a_1j+ra_2\Big),$$

as claimed. $\square$

Chapter 2

Weighted paths vs triangular recurrence relations

In this chapter, we will mainly study the recurrence relation (1) in the particular case where $b_{n,k} = 1$, and we replace the notation $a_{n,k}$ by $f_{n,k}$. Our goal is to define weighted paths that give a combinatorial interpretation of sequences of type (1), as was already done in Section 1.2.

Thus, unless otherwise stated, we will assume throughout this chapter that the relation (1) has the form (2.1):

$$F(n,k) = F(n-1,k-1) + f_{n,k}F(n-1,k),$$

and that the weights $f_{n,k}$ are given by

$$f_{n,k} = a_2 n + a_1 k + a_0.$$

We will also give a formula for the general case of sequences $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ of the form

$$T(n,k) = a_{n,k}T(n-1,k) + b_{n,k}T(n-1,k-1).$$

We will provide examples, in particular the r-Whitney numbers, and some identities that can be obtained by viewing $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ or $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ as families of weighted paths. In particular, any matrix $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ arising from a relation of the form (2.1) is a change-of-basis matrix.

We end this chapter by giving a simpler formula in the case $a_{n,k} = a_0 + a_1 k + a_2 n$ and $b_{n,k} = b_0 + b_1 k$.

2.1 Definition and notation

Our construction is based on the lattice $\mathbb{N}\times\mathbb{N}$ (see Figure 1.3).

Définition 2.1.1. *Let $(f_{i,j})_{i,j\in\mathbb{N}}$ be a family of real numbers. We define $\mathcal{R}_{n,k}$ as the set of all paths from $(0,0)$ to (n,k) in the lattice $\mathbb{N}\times\mathbb{N}$, using only:*

- *East steps **E** (i.e., $(i-1,j)\to(i,j)$), and*
- *North-East steps **NE** (i.e., $(i-1,j-1)\to(i,j)$).*

On $\mathcal{R}_{n,k}$, we define a weight function (valuation)

$$\mathrm{f} : \mathcal{R}_{n,k} \longrightarrow \mathbb{R},$$

such that each East step $(i-1, j) \to (i, j)$ receives the weight $f_{i,j}$.

We also denote by $F(n, k) = |\mathcal{R}_{n,k}|_{\mathrm{f}}$ the total weight of $\mathcal{R}_{n,k}$, and by

$$\mathbb{F} = (F(n, k))_{n,k\in\mathbb{N}}$$

the corresponding infinite matrix.

Remarque 2.1.1. We emphasize the distinction between $f_{n,k}$, the weight of a single step $(n-1, k) \to (n, k)$, and $F(n, k)$, the total weight of the entire set $\mathcal{R}_{n,k}$. Furthermore, the cardinality of $\mathcal{R}_{n,k}$ has already been discussed in 1.2 and equals $\binom{n}{k}$. Our focus here is its total weight. We will use the notation $\mathcal{F}(n, k)$ for the weighted set $(\mathcal{R}_{n,k}, \mathrm{f})$.

Remarque 2.1.2. Our construction is mainly designed to enumerate the case $b_{i,j} = 1$, but one may also assign weights to the North-East steps in order to treat the general situation. In that case, we denote the weight function by ω and consider the weighted set $(\mathcal{R}_{n,k}, \omega)$.

In some situations in this chapter, when the family $(f_{i,j})_{i,j\in\mathbb{N}}$ consists of positive integers, we may interpret $f_{i,j}$ as the number of possible colours for the East step $(i-1, j) \to (i, j)$. Under this interpretation, $\mathcal{F}(n, k)$ can be viewed simply as a finite combinatorial set.

Remarque 2.1.3. The definition allows us to derive the recurrence relation

$$\mathrm{F}(0, 0) = 1 \qquad \text{and} \qquad F(n, k) = \mathrm{F}(n-1, k-1) + f_{n,k}\mathrm{F}(n-1, k). \tag{2.1}$$

Conversely, any relation of this form, for $f_{n,k} \in \mathbb{R}$, can be interpreted as a weighting on $\mathcal{R}_{n,k}$. More generally, the triangular sequence define by

$$T(n, k) = a_{n,k}T(n-1, k) + b_{n,k}T(n-1, k-1),$$

is associated on weighing ω on $\mathcal{R}_{n,k}$ such that each East step $(i-1, j) \to (i, j)$ receives the weight $a_{i,j}$ and each North-East step $(i-1, j-1) \to (i, j)$ receives the weight $b_{i,j}$. Meanly $T(n, k)$ is the total weight $|\mathcal{R}_{n,k}|_\omega$.

Définition 2.1.2. *[**Definition of the r-Whitney numbers of the first and second kind $c_{m,r}(n, k)$ and $W_{m,r}(n, k)$ via recurrence**] For $m, r \in \mathbb{N}$,*

$$c_{m,r}(0, 0) = 1 \text{ and } c_{m,r}(n, k) = c_{m,r}(n-1, k-1) + (nm - m + r)c_{m,r}(n-1, k). \tag{2.2}$$

$$W_{m,r}(0, 0) = 1 \text{ and } W_{m,r}(n, k) = W_{m,r}(n-1, k-1) + (km + r)W_{m,r}(n-1, k). \tag{2.3}$$

2.2 Enumeration for $n > k$

We now derive formulas for $\Big(T(n, k)\Big)_{n,k\in\mathbb{N}}$ and $\Big(F(n, k)\Big)_{n,k\in\mathbb{N}}$, interpreting them as total weights.

Let $\mathcal{C}^{\uparrow}(k, n)$ denote the set of strictly increasing functions from $[k]$ to $[n]$.

Remarque 2.2.1. We observe that any path $\pi \in \mathcal{R}_{n,k}$ can be identified with a $\sigma \in \mathcal{C}^{\uparrow}(k,n)$ by taking $\sigma(i)$ to be the abscissa of the i-th North-East step **NE** of π (see for example Figure 2.1).

Now, to each $\sigma \in \mathcal{C}^{\uparrow}(k,n)$ we associate the strictly increasing list of the $n-k$ elements of $[n]$ that form the complement $[n] \setminus \sigma([k])$. We denote this image by $(\tilde{\sigma}_i)_{1 \leq i \leq n-k}$. Note that this correspondence is bijective.

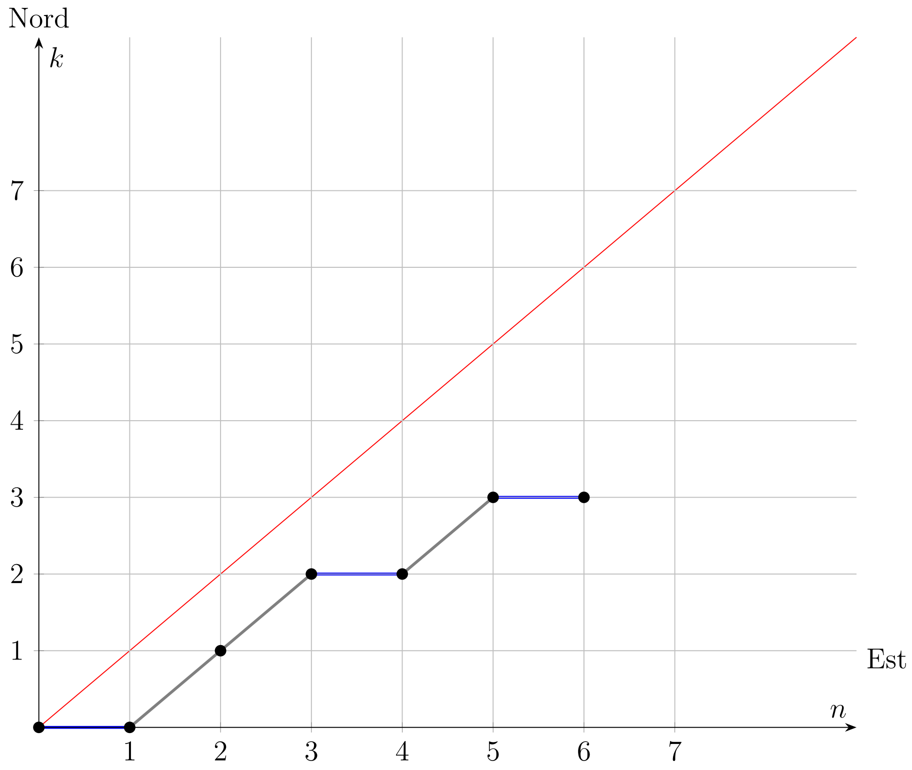


Figure 2.1: Path associated to $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}$. $\tilde{\sigma} = (1 \quad 4 \quad 6)$

Théorème 2.2.1. *Let* $\big(T(n,k)\big)_{n,k \in \mathbb{N}}$ *be a family satisfying the recurrence*

$$T(n,k) = a_{n,k} T(n-1,k) + b_{n,k} T(n-1,k-1),$$

with the usual initial conditions (2). *Then*

$$T(n,k) = \sum_{\sigma \in \mathcal{C}^{\uparrow}(k,n)} \prod_{i=1}^{n-k} \left(a_{\tilde{\sigma}_i, \tilde{\sigma}_i - i}\right) \prod_{i=1}^{k} \left(b_{\sigma(i), i}\right). \tag{2.4}$$

This relation is very similar to Theorem 1.3.1.

Proof. Recall our generalization at Remark 2.1.3 and our notation: $\omega : \mathcal{R}_{n,k} \to \mathbb{R}$ is the weighting, with weight $a_{i,j}$ assigned to each East step **E** $(i-1,j) \to (i,j)$ and weight $b_{i,j}$ to each North-East step **NE** $(i-1,j-1) \to (i,j)$. Since $T(n,k)$ is the total weight of $\mathcal{R}_{n,k}$, we have

$$T(n,k) = \sum_{\pi \in \mathcal{R}_{n,k}} \omega(\pi).$$

Using Remark 2.2.1, for each path $\pi \in \mathcal{R}_{n,k}$ we read off the weights of the East steps by viewing π as $\tilde{\sigma}$, and the weights of the North-East steps by viewing π as σ. Hence,

$$T(n,k) = \sum_{\sigma \in \mathcal{C}^{\uparrow}(k,n)} \prod_{i=1}^{n-k} \Big(a_{\tilde{\sigma}_i, \tilde{\sigma}_i - i}\Big) \prod_{i=1}^{k} \Big(b_{\sigma(i),i}\Big).$$

This proves the formula. $\square$

Corollaire 2.2.1. *Let $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ be sequence that satisfy the recurrence*

$$F(n,k) = F(n-1,k-1) + (a_2 n + a_1 k + a_0) F(n-1,k),$$

with the usual initial conditions. Then

$$F(n,k) = \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{i=1}^{n-k} \Big((a_2 + a_1) p_i - a_1 i + a_0\Big). \tag{2.5}$$

Corollaire 2.2.2. *We have*

$$c_{m,r}(n,k) = \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{j=1}^{n-k} \Big(p_j m - m + r\Big). \tag{2.6}$$

$$W_{m,r}(n,k) = \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{j=1}^{n-k} \Big((p_j - j) m + r\Big). \tag{2.7}$$

Let A, B, C be the sets defined by:

$$A = \{\bar{a} = (a_1, a_2, \cdots, a_n) : \sum_{i=1}^{n} a_i = n-k, \quad \forall i, a_i \in \{0,1\}\};$$
$$B = \{\tilde{b} = (b_1, b_2, \cdots, b_{n-k}) : 1 \le b_1 < b_2 < \cdots < b_{n-k} \le n\} \text{ and}$$
$$C = \{\underline{c} = (c_0, c_1, \cdots, c_k) : \sum_{i=0}^{k} c_i = n-k, \quad \forall i, c_i \in \{0,1,\cdots,n\}\}.$$

We equip these with the weights $\alpha, \beta, \beta', \delta$ given by

$$\alpha(\bar{a}) = \prod_{i=1}^{n} \Big(r+(i-1)m\Big)^{a_i}; \quad \beta(\tilde{b}) = \prod_{i=1}^{n-k} \Big(r+(b_i-1)m\Big); \quad \beta'(\tilde{b}) = \prod_{j=1}^{n-k} \Big(r+(b_j-j)m\Big) \text{ and } \delta(\underline{c}) = \prod_{i=0}^{k} \Big(r+im\Big)^{c_i}.$$

Proposition 2.2.1. *The correspondences*

(a) $\tilde{b} \mapsto \bar{a}$ *(where* $a_i = 1 \iff i \in \tilde{b}$*), and*

(b) $\tilde{b} \mapsto \underline{c}$ *(where* $c_i := \#\{j : b_j - j = i\}$*)*

define isomorphisms of weighted sets $(B, \beta) \cong (A, \alpha)$ *and* $(B, \beta') \cong (C, \delta)$*.*

Proof. For (a): the bijection is immediate. To see that it is also a morphism of weighted sets, observe that $a_i = 1$ if and only if $i \in \tilde{b}$, meaning that among the factors $\Big(r+(i-1)m\Big)^{a_i}$ contributing to the weight of $\bar{a}$:

- we get $\Big(r+(i-1)m\Big)$ if and only if $i = b_j$ for some j;
- we get $\Big(r+(i-1)m\Big)^0 = 1$ if and only if $i \neq b_j$ for all j.

Therefore, the weight of $\tilde{b}$ must be

$$\prod_{\{i:\, i\in\tilde{b}\}} \Big(r+(i-1)m\Big) = \prod_{x\in\tilde{b}} \Big(r+(x-1)m\Big),$$

as required. For (b), the map is well-defined since each $c_i \in \{0,1,\dots,n\}$ and $\sum c_i$ is exactly $|\tilde{b}| = n-k$. Its inverse is defined recursively as follows:

- for $i=0$: for all l with $1 \leq l \leq c_0$, set $b_l = l$;
- for i from 1 to k: for all l such that $\sum_{j=0}^{i-1} c_j < l \leq \sum_{j=0}^{i} c_j$, set $b_l = l+i$.

By construction, $b_1 \geq 1$ and $b_l < b_{l+1} \leq (n-k)+k = n$, so $\tilde{b} \in B$. It is also clear that this is the inverse of the correspondence. To verify that it is a morphism of weighted sets, it suffices to use the definition of c_i, which counts the indices j such that $b_j - j = i$. □

Thus we recover the explicit formulas for the r-Whitney numbers, which appear for instance in Cheon [7] and Wagner [34]:

Corollaire 2.2.3. *We have:*

$$c_{m,r}(n,k) = \sum_{\{a_1+a_2+\cdots+a_n=n-k,\, a_i=0,1\}} \prod_{i=1}^{n} \Big(r+(i-1)m)^{a_i}. \tag{2.8}$$

$$W_{m,r}(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k,\, 0\leq c_i\leq n\}} \prod_{i=0}^{k} \Big(r+im)^{c_i}. \tag{2.9}$$

Proof. This follows directly from Corollary 2.2.2 and Proposition 2.2.1. □

Définition 2.2.1. *[**Definition of the signed r-Whitney numbers $w_{m,r}(n,k)$ and the r-Lah numbers $WL(n,k)$ via recurrence**] For $m, r \in \mathbb{N}$,*

$$w_{m,r}(0,0) = 1 \text{ and } w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (-nm+m+-r)w_{m,r}(n-1,k). \tag{2.10}$$

$$WL_{m,r}(0,0) = 1 \text{ and } WL_{m,r}(n,k) = WL_{m,r}(n-1,k-1) + (km+nm-m+2r)WL_{m,r}(n-1,k). \tag{2.11}$$

Remarque 2.2.2. The number $w_{m,r}(n,k)$ differs from $c_{m,r}(n,k)$ by a global sign factor $(-1)^{n-k}$ coming from the $n-k$ East steps $\mathbf{E}$, so

$$w_{m,r}(n,k) = (-1)^{n-k} c_{m,r}(n,k).$$

2.3 On the product of the corresponding matrices

In this section, we highlight some situations where matrix products can be computed explicitly and will be useful in the remainder of this thesis. We focus in particular on combinatorial proofs.

Proposition 2.3.1. *Let $(x_{i,j})$ and $(y_{i,j})$ be two real sequences. Assume that*

$$\forall i,j,l: \ x_{l,i} = x_{l,j} = x_l, \quad y_{j,l} = y_{j,l} = y_l \ \Big((x_{n,k}) \textit{ is independent of } k \textit{ and } (y_{n,k}) \textit{ is independent of } n\Big). \tag{2.12}$$

Then $\mathbb{X}\mathbb{Y} = \Big((Z(n,k))\Big)_{n,k\in\mathbb{N}}$, *where* $Z(n,k) = |\mathcal{R}_{n,k}|_{\mathrm{z}}$ *and the weights satisfy* $z_{n,k} = x_{n,k} + y_{n,k}$.

Proof. The first part of the proof deals with the case where the weights are positive integers, so we can interpret them as colours.

First, we fix the notation: $\nearrow$ denotes a North–East step **NE**; $\overset{p}{\nearrow}$ denotes a p-tuple of p consecutive North–East steps **NE**; and $\xrightarrow{\mathbf{c}}$ denotes an East step **E** $(i-1,j) \to (i,j)$ of colour $\mathbf{c}$ (so $1 \le \mathbf{c} \le f_{i,j}$). In other words, $\mathbf{c}$ indicates the index of the colour of $\xrightarrow{\mathbf{c}}$ among the $f_{i,j}$ possible colours of this East step **E** $(i-1,j) \to (i,j)$. Moreover, if u is a sequence of steps (or a path), we define $u + \nearrow$, $u + \overset{p}{\nearrow}$ and $u + \xrightarrow{\mathbf{c}}$ to be the path obtained by concatenating u with one of these steps at the end. Similarly, $\nearrow + u$, $\overset{p}{\nearrow} + u$ and $\xrightarrow{\mathbf{c}} + u$ denote the corresponding concatenations at the beginning.
Now let $\mathcal{Z}(n,k)$ be the set of paths of this type with colours given by $z_{n,k} = x_{n,k} + y_{n,k}$, and set

$$\mathcal{E}(n,k) = \bigcup_{l\ge 0} \mathcal{X}(n,l) \times \mathcal{Y}(l,k).$$

We now show that there is a bijection between $\mathcal{Z}(n,k)$ and $\mathcal{E}(n,k)$.
By assumption, $x_{i,j} = x_i$ and $y_{i,j} = y_j$.
Let $w = w_1 w_2 \cdots w_n \in \mathcal{Z}(n,k)$. We start by setting $u^{(0)} = v^{(0)} = \emptyset$.
We then construct $(u^{(n)}, v^{(n)})$, the image of w, recursively.

For i ranging from 1 to n:

- Case 1: If $w_i = \nearrow$. Set $u^{(i)} = u^{(i-1)} + \nearrow$ and $v^{(i)} = v^{(i-1)} + \nearrow$;
- Case 2: If $w_i = \xrightarrow{\mathbf{c}}$ with $c \le x_i$. Set $u^{(i)} = u^{(i-1)} + \xrightarrow{\mathbf{c}}$ and $v^{(i)} = v^{(i-1)}$;
- Case 3: If $w_i = \xrightarrow{\mathbf{c}}$ with $\mathbf{c} > x_i$. Set $u^{(i)} = u^{(i-1)} + \nearrow$ and $v^{(i)} = v^{(i-1)} + \xrightarrow{\mathbf{c}\text{-}x_i}$.

For the compatibility of Cases 1 and 2, this is clear. Case 3 is also compatible, because if $\mathbf{c} = z_{i,j}$, then j is at most equal to the height of $w^{(i-1)} + 1$, which is also the height of $v^{(i-1)} + 1$ (by Case 1). Hence $\mathbf{c} - x_i \le$ the height of $v^{(i-1)} + 1$.
At each step, we add one extra step to u, so u has the same length as w. Moreover, $v^{(i-1)}$ goes to $v^{(i)}$ with an increase in height if and only if $w_i = \nearrow$, so v has the same height as w. By the way we construct u and v in all three cases, $u^{(i-1)}$ goes to $u^{(i)}$ with an increase in height if and

only if $v^{(i-1)}$ goes to $v^{(i)}$ with an increase in length. Thus $(u,v) \in \mathcal{X}(n,l) \times \mathcal{Y}(l,k)$ for some $l = k, \dots, n$, and the correspondence is well-defined.

This correspondence $w \mapsto (u^{(n)}, v^{(n)})$ is a bijection from $\mathcal{Z}(n,k)$ to $\mathcal{E}(n,k)$. Indeed, its inverse can be defined recursively as follows. Let $(u,v) \in \mathcal{X}(n,l) \times \mathcal{Y}(l,k)$ for some $l = k, k+1, \dots, n-1, n$.
Set $w^{(0)} = \emptyset$, $j = 1$, and for i ranging from 1 to n:

- Case 1: $u_i = \xrightarrow{\mathbf{c}}$. Set $w^{(i)} = w^{(i-1)} + \xrightarrow{\mathbf{c}}$;
- Case 2: $u_i = \nearrow$ and $v_j = \xrightarrow{\mathbf{c}}$. Set $w^{(i)} = w^{(i-1)} + \xrightarrow{\mathbf{c}+x_i}$; $j = j+1$;
- Case 3: $u_i = \nearrow$ and $v_j = \nearrow$. Set $w^{(i)} = w^{(i-1)} + \nearrow$; $j = j+1$.

The inverse image of (u,v) is therefore $w^{(n)}$. First, we require one step in v each time u_i increases in height (cases 2 and 3). Hence, calling v_j at each step $\nearrow$ is fully justified. Moreover, by construction, w has length n. Next, the transition from $w^{(i-1)}$ to $w^{(i)}$ increases its height if and only if v also increases its height at the unique step j (case 3); therefore w and v have the same height. It follows that the inverse map is well defined.

Remarque 2.3.1 (Non-integer case)**.** Up to this point we have been reasoning under the assumption that $f_{n,k}$ is the number of possible colours, and hence is always a positive integer. However, if we instead interpret $|f_{n,k}|$ as the number of colours and weight each East step **E** $(i-1,j) \to (i,j)$ by -1, then the formula obtained in Corollary 2.2.1 remains valid.

The general case is then obtained by polynomial extension. □

Corollaire 2.3.1. *If we denote by* $\mathbb{W}_1'$ *and* $\mathbb{W}_2$ *the matrices* $(w_{m,r}(n,k))$ *and* $(W_{m,r}(n,k))$ *respectively, then*

$$\mathbb{W}_1'\mathbb{W}_2 = \mathbb{I}.$$

Proof. The matrix $\mathbb{W}_1'\mathbb{W}_2$ is the matrix associated with paths of this type whose weights are

$$f_{n,k} = -nm + m - r + km + r = m(k-n+1).$$

Applying Corollary 2.2.1, we obtain

$$\Big(\mathbb{W}_1'\mathbb{W}_2\Big)(n,k) = \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{i=1}^{n-k} \Big((1-i)m\Big) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

□

Remarque 2.3.2. Thus the r-Lah numbers defined in Definition 2.2.1 by (2.11) can also be given by

$$\Big(WL_{m,r}(n,k)\Big)_{n,k \in \mathbb{N}} = \mathbb{W}_3 = \mathbb{W}_1\mathbb{W}_2,$$

if we denote $\Big(w_{m,r}(n,k)\Big)$ by $\mathbb{W}_1$. Indeed, the matrix $\mathbb{W}_1\mathbb{W}_2$ is the matrix associated with paths of this type whose weights are

$$f_{n,k} = nm - m + r + km + r = 2r + m(n+k-1).$$

Proposition 2.3.2. *Let $\mathcal{X}(n,k)$ and $\mathcal{Y}(n,k)$ be two sets of weighted paths of this type, given by the weights (colours) $(x_{n,k})$ and $(y_{n,k})$ respectively. The sequences $X(n,k)$ and $Y(n,k)$ are represented by the matrices $\mathbb{X}$ and $\mathbb{Y}$. Suppose that*

- *for all i,j,l we have $x_{l,i} = x_{l,j} = x_l$ and $y_{j,l} = y_{j,l} = y_l$ (that is, $(x_{n,k})$ is independent of k and $(y_{n,k})$ is independent of n),*
- *and $x_{l+1} = y_l$.*

Then we have, for all n,k,

$$\mathbb{Y}\,\mathbb{X}' = I,$$

where $\mathbb{X}'$ is defined by $X'(n,k) = (-1)^{(n-k)}X(n,k)$.

Proof. We first prove the statement in the case where the weights are positive integers, so that we can interpret them as colours.
We keep the previous notations and set

$$\mathcal{Z}(n,k) = \bigcup_{i\geq 0} \mathcal{Y}(n,i) \times \mathcal{X}(i,k).$$

Define

$$w(u,v) = (-1)^{i+k}$$

for $(u,v) \in \mathcal{Y}(n,i) \times \mathcal{X}(i,k)$, and consider the map

$$(u' + \xrightarrow{\mathbf{c}} + \overset{p}{\nearrow}, v' + \overset{p}{\nearrow}) \mapsto (u' + \nearrow + \overset{p}{\nearrow}, v' + \xrightarrow{\mathbf{c}} + \overset{p}{\nearrow}),$$
$$(u' + \nearrow + \overset{p}{\nearrow}, v' + \xrightarrow{\mathbf{c}} + \overset{p}{\nearrow}) \mapsto (u' + \xrightarrow{\mathbf{c}} + \overset{p}{\nearrow}, v' + \overset{p}{\nearrow}).$$

This map is a weighted involution of $(\mathcal{Z}(n,k), w)$ whose set of fixed points is

$$Fix = \begin{cases} \{\overset{p}{\nearrow}\} & \text{if } n = k, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence

$$\left(\mathbb{Y}\mathbb{X}'\right)(n,k) = |\mathcal{Z}(n,k)|_w = |Fix|_w = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise,} \end{cases}$$

and the result follows. The general (non-integer) case is then obtained by polynomial extension. □

We also observe that this involution gives a bijection between

$$\bigcup_{i\geq 0} \mathcal{Y}(n,2i) \times \mathcal{X}(2i,k) \quad \text{and} \quad \bigcup_{i\geq 0} \mathcal{Y}(n,2i+1) \times \mathcal{X}(2i+1,k)$$

when $n > k$, which explains the identity

$$\sum_{i\geq 0} \mathrm{Y}(n,i)(-1)^{i-k}\mathrm{X}(i,k) = 0.$$

We can reformulate this proposition as follows.

Remarque 2.3.3. Let $\mathcal{F}(n,k)$ and $\mathcal{Y}(n,k)$ be two sets of paths of this type, with weights (colours) $(f_{n,k})$ and $(y_{n,k})$ respectively. The corresponding sequences $F(n,k)$ and $Y(n,k)$ are represented by the matrices $\mathbb{F}$ and $\mathbb{Y}$. If for all i,j,l we have $f_{l,i} = f_{l,j} = f_l$ and $y_{j,l} = y_{j,l} = y_l$ (so $(f_{n,k})$ is independent of k, whereas $(y_{n,k})$ is independent of n), and if $f_{l+1} = -y_l$, then for all n,k we have $\mathbb{YF} = I$.

Corollaire 2.3.2. *If we denote by $\mathbb{W}_1$ and $\mathbb{W}_2$ the matrices $(c_{m,r}(n,k))$ and $(W_{m,r}(n,k))$ respectively, then*

$$\mathbb{W}_2\mathbb{W}_1' = \mathbb{I}.$$

2.4 Enumerating polynomials for $c_{m,r}(n,k)$ by height and $W_{m,r}(n,k)$ by length

We also recover the following two rational generating functions found in the literature (see, for example, Cheon [7]); here we present them using our current approach.

Proposition 2.4.1. *We have*

$$\sum_{k\geq 0} c_{m,r}(n,k)x^k = \prod_{i=1}^{n}(x - m + r + im).$$

Proof. Let $\mathcal{C}_n$ be the set of paths corresponding to the numbers $(c_{m,r}(n,k))$ of length n, and define $w(\pi) = x^{h(\pi)}$ for each $\pi \in \mathcal{C}_n$, where $h(\pi)$ is the height of π. To obtain a path π of length n, there are only two possibilities starting from a path of length $n-1$:

- either the last step is North-East, so $\pi = u + \nearrow$ and $h(\pi) = h(u) + 1$;
- or the last step is East, so $\pi = u + \longrightarrow$ and $h(\pi) = h(u)$, but there are $nm - m + r$ possible colours for this step.

Hence

$$\begin{aligned}
|\mathcal{C}_n|_w &= \sum_{\pi\in\mathcal{C}_n} w(\pi) = \sum_{\pi\in\mathcal{C}_n} x^{h(\pi)} \\
&= \sum_{u\in\mathcal{C}_{n-1}} \left(x^{h(u)+1} + (r + nm - m)x^{h(u)}\right) \\
&= (x + r + nm - m)\,|\mathcal{C}_{n-1}|_w.
\end{aligned}$$

Since $|\mathcal{C}_0|_w = 1$, the result follows by induction. $\square$

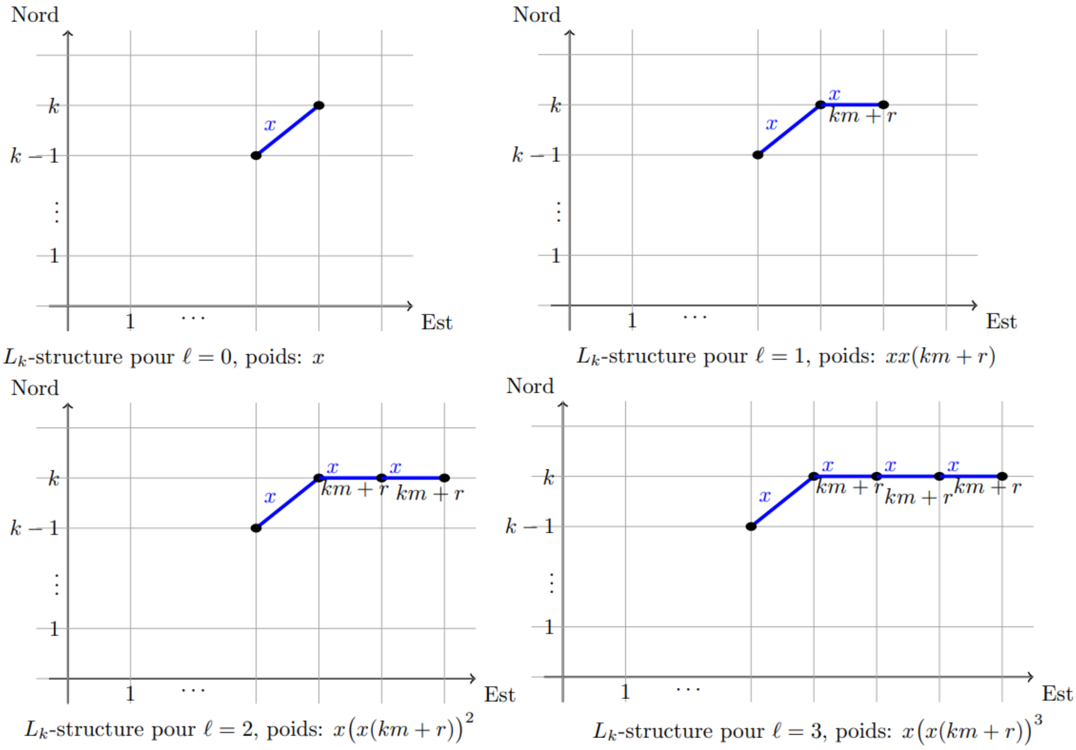
Proposition 2.4.2. *We have*

$$\sum_{n\geq 0} W_{m,r}(n,k)x^n = \prod_{i=0}^{k} \frac{x}{1-(r+mi)x}.$$

Proof. Let $\mathcal{H}_k$ denote the class of paths corresponding to $(W_{m,r}(n,k))_{n\geq 0}$ of height k. Its generating function is

$$\mathcal{H}_k(x) = \sum_{n\geq 0} W_{m,r}(n,k)x^n.$$

Consider another combinatorial class $\mathcal{L}_k$, consisting of a pair of steps: one North-East step from height $k-1$ to k, followed by an l-step sequence of East steps (a horizontal run) for some $l \in \mathbb{N}$.



L_k-structure pour $\ell = 0$, poids: x

L_k-structure pour $\ell = 1$, poids: $xx(km+r)$

L_k-structure pour $\ell = 2$, poids: $x\big(x(km+r)\big)^2$

L_k-structure pour $\ell = 3$, poids: $x\big(x(km+r)\big)^3$

Figure 2.2: Examples of $\mathcal{L}_k$-structures .

We observe that each $\mathcal{H}_k$-structure can be decomposed as a pair (H, L) where H is an $\mathcal{H}_{k-1}$-structure and L is an $\mathcal{L}_k$-structure. Thus

$$\mathcal{H}_k = \mathcal{H}_{k-1} \times \mathcal{L}_k \quad \text{and} \quad \mathcal{H}_k(x) = \mathcal{H}_{k-1}(x) \times \mathcal{L}_k(x),$$

with

$$\mathcal{L}_k(x) = \sum_{l\geq 0} x\Big(x(km+r)\Big)^l = \frac{x}{1-(km+r)x}.$$

Since $\mathcal{H}_0(x) = \frac{1}{1-rx}$, the stated product formula follows. $\square$

2.5 Definition of $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ as a change-of-basis matrix

Several authors characterize the r-Whitney numbers as a change-of-basis matrix in the infinite-dimensional vector space of polynomials $\mathbb{K}[x]$. In this section, we show that these identities can be recovered using our present approach, and then generalized. We also use weighted paths to express any sequence of the form (2.1) as a change-of-basis matrix.

Proposition 2.5.1. *We have*

$$(mx+r)^n = \sum_{k\geq 0} W_{m,r}(n,k)\, m^k\, (x)^{(\underline{k})},$$

where $(x)^{(\underline{k})} = x(x-1)\cdots(x-k+1)$.

Proof. Let $\mathcal{D}_n$ be the set of paths that enumerate the numbers $\Big(W_{m,r}(n,k)\Big)_{k\in\mathbb{N}}$ of length n, and for each $\pi \in \mathcal{D}_n$ define

$$w(\pi) = m^{h(\pi)}(x)^{(\underline{h(\pi)})},$$

where $h(\pi)$ is the height of π. Then

$$|\mathcal{D}_n|_w = \sum_{\pi\in\mathcal{D}_n} m^{h(\pi)}(x)^{(\underline{h(\pi)})} = \sum_{k\geq 0}\sum_{\pi\in\mathcal{D}_{n,k}} m^k(x)^{(\underline{k})} = \sum_{k\geq 0} W_{m,r}(n,k)m^k(x)^{(\underline{k})}.$$

Considering the two ways to go from length $n-1$ to n, we get

$$\begin{aligned}
|\mathcal{D}_n|_w &= \sum_{\pi\in\mathcal{D}_n} m^{h(\pi)}(x)^{(\underline{h(\pi)})}\\
&= \sum_{u\in\mathcal{D}_{n-1}} \Big(m\Big(x-(h(u)+1)\Big)w(u) + \Big((h(u)+1)m+r\Big)w(u)\Big)\\
&= (mx+r)\,|\mathcal{D}_{n-1}|_w.
\end{aligned}$$

Since $|\mathcal{D}_0|_w = 1$, the result follows by induction. $\square$

Recall that for $m\neq 0$, the sequence $\Big((mx+r)^n\Big)_{n\geq 0}$ forms a basis of $\mathbb{K}[x]$. The same holds for $\Big((x)^{(\underline{k})}\Big)_{k\geq 0}$. Thus Proposition 2.5.1 says that $(W_{m,r}(n,k))$ is the change-of-basis matrix from $\Big((mx+r)^n\Big)_{n\geq 0}$ to $\Big((x)^{(\underline{k})}\Big)_{k\geq 0}$.

Proposition 2.5.2. *We have*

$$m^n(x)^{(\underline{n})} = \sum_{k\geq 0} w_{m,r}(n,k)\,(mx+r)^k.$$

Proof. Proposition 2.4.1 also gives

$$\sum_{k\geq 0} w_{m,r}(n,k)x^k = \prod_{k=1}^{n}(x+m-r-km).$$

Replacing x by $mx+r$ yields the desired identity. $\square$

These results motivate us to do something similar for a general sequence $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$. We can obtain various relations depending on suitable choices of weights.

Théorème 2.5.1. *If* $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ *satisfies* (2.1)*, then*

$$(x|a_2)^{(\overline{n})} = \sum_{k=0}^{n} F(n,k)\,(x-a_0-a_2|a_1)^{(\underline{k})}, \tag{2.13}$$

where

$$(x|a_2)^{(\overline{n})} = \prod_{i=0}^{n-1}(x+ia_2) \quad \textit{and} \quad (x-a_0-a_2|a_1)^{(\underline{k})} = \prod_{i=0}^{k-1}(x-a_0-a_2-ia_1).$$

Proof. Let $\mathcal{E}_n$ be the set of paths that enumerate the numbers $\Big(F(n,k)\Big)_{k\in\mathbb{N}}$ of length n, and for each $\pi\in\mathcal{E}_n$ define

$$v(\pi) = (x-a_0-a_2|a_1)^{(\underline{h(\pi)})},$$

where $h(\pi)$ is the height of π. Then

$$|\mathcal{E}_n|_v = \sum_{\pi\in\mathcal{E}_n}(x-a_0-a_2|a_1)^{(\underline{h(\pi)})} = \sum_{k\geq 0}\sum_{\pi\in\mathcal{E}_{n,k}}(x-a_0-a_2|a_1)^{(\underline{k})} = \sum_{k\geq 0}F(n,k)\,(x-a_0-a_2|a_1)^{(\underline{k})}.$$

Considering the two ways to go from length $n-1$ to n, we obtain

$$\begin{aligned}|\mathcal{E}_n|_v &= \sum_{\pi\in\mathcal{E}_n}(x-a_0-a_2|a_1)^{(\underline{h(\pi)})}\\ &= \sum_{u\in\mathcal{E}_{n-1}}\Big(\Big(x-a_0-a_2-a_1(h(u)+1)\Big)\,v(u) + \Big(a_2 n + a_1(h(u)+1)+a_0\Big)\,v(u)\Big)\\ &= (x+a_2(n-1))\,|\mathcal{E}_{n-1}|_v.\end{aligned}$$

Since $|\mathcal{E}_0|_v = 1$, the identity follows by induction. □

We also note that the sequences $\Big((x|a_2)^{(\overline{n})}\Big)_{n\geq 0}$ and $\Big((x-a_0-a_2|a_1)^{(\underline{k})}\Big)_{k\geq 0}$ are both bases of $\mathbb{K}[x]$. Thus Theorem 2.5.1 shows that $(F(n,k))$ is the change-of-basis matrix from $\Big((x|a_2)^{(\overline{n})}\Big)_{n\geq 0}$ to $\Big((x-a_0-a_2|a_1)^{(\underline{k})}\Big)_{k\geq 0}$. We observe that this relation is connected to generalized Stirling numbers. Such generalizations are discussed in Hsu [35] and also in Maier [31].

2.6 Combinatorial interpretation of $m^k k!\,W_{m,r}(n,k)$

Let $E_{m,r}(k)$ be the set consisting of r uncoloured elements $e_1, e_2, \ldots, e_r$ and km coloured elements with k colours and m types: $c_{i,j}$ is the element of colour i and type j. Define

$$A_{m,r}(n,k) = \Big\{f:[n]\to E_{m,r}(k)\ :\ \forall i,\ \exists j \text{ such that } c_{i,j}\in f([n])\Big\}.$$

Proposition 2.6.1. *The explicit formula for* $a_{m,r}$ *is*

$$a_{m,r}(n,k) = |A_{m,r}(n,k)| = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(mj+r)^n. \tag{2.14}$$

Before proving this, recall the principle of inclusion–exclusion: if $E = \bigcup_{i=1}^{k} A_i$, then

$$\Big|\bigcap_{i=1}^{k} A_i^c\Big| = |E| + \sum_{j=1}^{k}(-1)^j \sum_{1\le i_1<i_2<\cdots<i_j\le k} \Big|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_j}\Big|.$$

Proof. For $i \in [k]$, set

$$A_i := \Big\{f : [n] \to E_{m,r}(k) \;:\; \forall j,\; c_{i,j} \notin f([n])\Big\}.$$

Clearly,

$$A_{m,r}(n,k) = \bigcap_{i=1}^{k} A_i^c.$$

On the other hand, for any $I \subseteq [k]$, an element of $\bigcap_{i\in I} A_i$ is a map from $[n]$ into

$$E_{m,r}(k) \setminus \{c_{i,j} : j \in [m],\ i \in I\}.$$

Hence, if $|I| = j$, we have

$$\Big|\bigcap_{i\in I} A_i\Big| = \Big((k-j)m + r\Big)^n.$$

Applying inclusion–exclusion yields (2.14). □

Proposition 2.6.2. *The sequence* $\Big(a_{m,r}(n,k)\Big)_{n,k\in\mathbb{N}}$ *satisfies the recurrence*

$$a_{m,r}(0,0) = 1, \quad a_{m,r}(n,k) = 0 \text{ if } n<k, \quad a_{m,r}(n,k) = km\, a_{m,r}(n-1,k-1) + (km+r)\, a_{m,r}(n-1,k). \tag{2.15}$$

Proof. The correspondence between $A_{m,r}(n,\cdot)$ and $A_{m,r}(n-1,\cdot)$ can be summarized as follows (let $f \in A_{m,r}(n,k)$):

- Case 1: $f(n)$ is the only element of its colour among the elements of $f([n])$. We may remove this colour without losing information, which identifies f with an element of $A_{m,r}(n-1,k-1)$.
  Conversely, to go from $A_{m,r}(n-1,k-1)$ to $A_{m,r}(n,k)$, we must introduce a new colour. There are k choices for the colour and m choices for the type of $f(n)$, hence km possibilities.

- Case 2: $f(n)$ is not coloured. Then f corresponds to an element of $A_{m,r}(n-1,k)$.
  Conversely, we have r choices for $f(n)$.

- Case 3: $f(n)$ is coloured but not the only element of its colour. In this case, we remove only the point n (and keep the colour), which yields a map in $A_{m,r}(n-1,k)$.
  Conversely, in this situation there are km choices for $f(n)$ (the coloured elements).

This yields the stated recurrence. □

Proposition 2.6.3. *We have*

$$a_{m,r}(n,k) = m^k k!\, W_{m,r}(n,k).$$

Proof. One can prove this analytically. Here, however, we give a justification in terms of weighted paths. The recurrence (2.15) shows that $a_{m,r}(n,k)$ counts almost the same paths[1] as $W_{m,r}(n,k)$, except that for $a_{m,r}(n,k)$, each North-East step also has k colour choices at that level. This contributes a factor $m \times 2m \times \cdots \times km = k!\, m^k$. $\square$

Corollaire 2.6.1. *For $m \neq 0$,*

$$W_{m,r}(n,k) = \frac{1}{k!m^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj+r)^n.$$

Proof. This follows directly from Propositions 2.6.1 and 2.6.3. $\square$

Remarque 2.6.1. If $f_{n,k}$ in (2.1) is independent of n and of the form $f_{n,k} = a_0 + a_1 k + a_2 n$ with $a_2 = 0$ and $a_1 \neq 0$, then

$$F(n,k) = \frac{1}{k!a_1^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (a_1 j + a_0)^n.$$

Remarque 2.6.2. The approach in the proof of Proposition 2.6.3 also shows that if $\big(T(n,k)\big)_{n,k\in\mathbb{N}}$ satisfies (1) with $a_{i,j} = a_0 + a_1 j + a_2 i$ and $b_{i,j} = b_0 + b_1 j$, then

$$T(n,k) = (b_0 + b_1 | b_1)^{(\overline{k})}\, F(n,k).$$

Moreover, Proposition 2.3.1 shows that $\big(F(n,k)\big)_{n,k\in\mathbb{N}}$ is the product of the matrix $\big(c_{a_2,0}(n,k)\big)_{n,k\in\mathbb{N}}$ with $\big(W_{a_1,a_0+a_2}(n,k)\big)_{n,k\in\mathbb{N}}$. Note that the numbers $\big(c_{a_2,0}(n,k)\big)_{n,k\in\mathbb{N}}$ satisfy a recurrence similar to that of the Stirling numbers of the first kind $\big(s(n,k)\big)_{n,k\in\mathbb{N}}$; the weight of each East step $(n-1,k) \to (n,k)$ becomes $a_2(n-1)$ instead of $n-1$ for $\big(s(n,k)\big)$. Hence $c_{a_2,0}(n,k) = a_2^{n-k} s(n,k)$ and

$$T(n,k) = (b_0 + b_1 | b_1)^{(\overline{k})2} \sum_{i\geq 0} a_2^{n-i} s(n,i)\, W_{a_1,a_0+a_2}(i,k). \tag{2.16}$$

This relation will be discussed again in Proposition 3.6.1. In that section we give another proof of the explicit formula found in Neuwirth [2].

[1] Assuming m, r are integers and interpreting the weights as numbers of colours.

[2] Here $(b_0 + b_1 | b_1)^{(\overline{k})} = \prod_{i=0}^{k-1} (b_0 + b_1 + i b_1)$.

Chapter 3

Triangular recurrence vs exponential Riordan matrix

In this part, we study sequences of the form (1)

$$T(n,k) = (a_2n + a_1k + a_0)T(n-1,k) + (b_2n + b_1k + b_0)T(n-1,k-1),$$

and in particular of the form (2.1)

$$F(n,k) = (a_2n + a_1k + a_0)F(n-1,k) + F(n-1,k-1),$$

using exponential Riordan matrices. In this chapter, we construct the Riordan matrix associated with these sequences and use this result to obtain an explicit expression for the sequence. The case (2.1) is treated first and we use it to derive their explicit formula, as well as a formula for the general case (1) when $b_2 = 0$. We conclude this chapter by also discussing the limitations of this approach. To simplify the terminology, we will simply use the word Riordan matrix to refer to exponential Riordan matrices. Our main reference for Riordan matrices is Sprugnoli [36].

3.1 Generalities on Riordan matrices

Définition 3.1.1 ([7]). *Let f and g be two formal power series in the ring $\mathbb{C}[[z]]$ such that*

$$g(0) \neq 0, \quad f(0) = 0, \quad f'(0) \neq 0. \tag{3.1}$$

A Riordan matrix associated with the pair (g, f) is a matrix $\mathbb{F} = (F(n,k)_{n,k\in\mathbb{N}})$ defined by

$$F(n,k) = [\tfrac{z^n}{n!}]g(z)\frac{(f(z))^k}{k!}. \tag{3.2}$$

In this case, we write $\mathbb{F} =< g, f >$. Sometimes, one starts from an infinite array $\left(L_{n,k}\right)_{n,k\in\mathbb{N}}$ and looks for g, f satisfying this definition. A Riordan matrix can also be regarded as a pair $< g, f >$ satisfying (3.1). What makes this theory particularly important is that there is an internal composition law satisfying several algebraic properties.

Proposition 3.1.1. *The map from $\mathbb{C}[[z]] \times \mathbb{C}[[z]]$ to $\mathbb{C}[[z]]$ defined by*

$$< g, f > \star < h, l > = < g\Big(h \circ f\Big), l \circ f >$$

defines an internal composition law on the set of Riordan matrices. Moreover, this set together with the operation $\star$ forms a group, called the Riordan group.

Proof. Associativity is straightforward.
Next, one easily checks that

$$\begin{aligned}
&< g, f > \star < 1, z > = < g, f > = < 1, z > \star < g, f >; \\
&< g, f > \star < \frac{1}{g \circ f^{-1}}, f^{-1} > = < 1, z > .
\end{aligned}$$

□

Proposition 3.1.2. *This composition law coincides with the usual matrix product.*

Proof. Let $(L(n,k))_{n,k \in \mathbb{N}} = < g, f >$ and $(M(n,k))_{n,k \in \mathbb{N}} = < h, l >$ be two Riordan matrices. Let $(F(n,k))_{n,k \in \mathbb{N}}$ denote their product. Then

$$\begin{aligned}
\sum_{n \in \mathbb{N}} F(n,k) \frac{z^n}{n!} &= \sum_{n \in \mathbb{N}} \Big(\sum_{j \in \mathbb{N}} \mathrm{L}(n,j) \mathrm{M}(j,k) \Big) \frac{z^n}{n!} = \sum_{j \in \mathbb{N}} \mathrm{M}(j,k) \Big(\sum_{n \in \mathbb{N}} \mathrm{L}(n,j) \frac{z^n}{n!} \Big) \\
&= \sum_{j \in \mathbb{N}} \mathrm{M}(j,k) g(z) \frac{(f(z))^j}{j!} = g(z) \sum_{j \in \mathbb{N}} \mathrm{M}(j,k) \frac{(f(z))^j}{j!} \\
&= g(z) h \circ f(z) \frac{\Big(l \circ f(z)\Big)^k}{k!}.
\end{aligned}$$

□

Proposition 3.1.3. *Let $L = < g, f >$ and $V = (v_0, v_1, v_2, ...)^T$ be an (infinite) column vector whose exponential generating function is $v(x) = \sum_{n \in \mathbb{N}} v_n \frac{z^n}{n!}$. Then*

$$\Big(< g, f > V \Big)_n = \sum_{k \in \mathbb{N}} L(n,k) v_k = [\frac{z^n}{n!}] g(z) v \circ f(z).$$

This is the fundamental property of a Riordan matrix.

Proof.

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \Big(\sum_{k \in \mathbb{N}} L(n,k) v_k \Big) \frac{z^n}{n!} &= \sum_{k \in \mathbb{N}} v_k \Big(\sum_{n \in \mathbb{N}} L(n,k) \frac{z^n}{n!} \Big) \\
&= \sum_{k \in \mathbb{N}} v_k g(z) \frac{\Big(f(z)\Big)^k}{k!} \\
&= g(z) v(f(z)).
\end{aligned}$$

□

Some properties

Lemme 3.1.1. *Any Riordan matrix* $< g, f >$ *can be factorized as*

$$< g, f > = < g, z > \star < 1, f > . \tag{3.3}$$

This looks obvious, but it is very useful.

Lemme 3.1.2. *A Riordan matrix* $< g, f >$ *can be factorized as*

$$< g, f > = < g, z > \cdot \Big(A(n,k)\Big)_{n,k\in\mathbb{N}},$$

where $A(0,0) = 1$ *and* $A(n,k) = \mathrm{F}(n-1, k-1)$ *with* $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}} = < f', f >$.

This is lemma 3.1 of Cheon [7], where it is pointed out that $\mathbb{A} = \mathbb{I}_1 \oplus < f', f >$, with $\mathbb{I}_j$ the identity matrix of order j, and if $\mathbb{M}_1 \in \mathcal{M}_p(\mathbb{R})$ and $\mathbb{M}_2 \in \mathcal{M}_q(\mathbb{R})$,

$$\mathbb{M}_1 \oplus \mathbb{M}_2 = \begin{pmatrix} \mathbb{M}_1 & 0_{p\times q} \\ 0_{q\times p} & \mathbb{M}_2 \end{pmatrix}$$ [1].

Proof. In addition to equation (3.3), we also have

$$\begin{aligned}
\frac{\Big(f(z)\Big)^k}{k!} &= \int f'(z) \frac{(f(z))^{k-1}}{(k-1)!}\, dz \\
&= \int \sum_{n\in\mathbb{N}} \mathrm{F}(n, k-1) \frac{z^n}{n!}\, dz \\
&= \sum_{n\in\mathbb{N}} \mathrm{F}(n, k-1) \frac{z^{n+1}}{(n+1)!} + \frac{(f(0))^k}{k!} \\
&= \sum_{n\geq 1} \mathrm{F}(n-1, k-1) \frac{z^n}{n!} + \frac{(f(0))^k}{k!}.
\end{aligned}$$

□

From this lemma, we can conclude the following.

Corollaire 3.1.1. *If* $g(z) = \sum_{n\in\mathbb{N}} g_n \frac{z^n}{n!}$ *and* $f(z) = \sum_{n\in\mathbb{N}} f_n \frac{z^n}{n!}$, *then*

$$< g, f > = \Big(G(n,k)\Big)_{n,k\in\mathbb{N}} \prod_{j\geq 1} \Big(\mathbb{I}_j \oplus \Big(T(n,k)\Big)_{n,k\in\mathbb{N}}\Big),$$

where $G(n,k) = \binom{n}{k} g_{n-k}$ *and* $T(n,k) = \binom{n}{k} f_{n+1-k}$. *In particular,*

$$< f', f > = \prod_{j\in\mathbb{N}} \Big(\mathbb{I}_j \oplus \Big(T(n,k)\Big)_{n,k\in\mathbb{N}}\Big).$$

[1] $0_{i\times j}$ is a zero matrix with i rows and j columns.

Proof. First, we have

$$g(z)\frac{z^k}{k!} = \sum_{n\in\mathbb{N}} g_n \frac{z^{n+k}}{n!k!} = \sum_{n\geq k} \binom{n}{k} g_{n-k}\frac{z^n}{n!} = \sum_{n\geq k} G(n,k)\frac{z^n}{n!}.$$

Hence $< g, z >= \Big(G(n,k)\Big)_{n,k\in\mathbb{N}}$.
Secondly, we have $< f', f >=< f', z > \cdot\Big([1]\oplus < f', f >\Big)$ and $< f', z >= \mathbb{T}$. Indeed,

$$f'(z)\frac{z^k}{k!} = \Big(\sum_{n\geq 1} f_n \frac{z^{n-1}}{(n-1)!}\Big)\frac{z^k}{k!} = \sum_{n\geq k}\binom{n}{k} f_{n+1-k}\frac{z^n}{n!} = \sum_{n\geq k} T(n,k)\frac{z^n}{n!}.$$

□

In the next section, we investigate the possibility of writing a Riordan matrix for sequences $(F(n,k))_{n,k\in\mathbb{N}}$ satisfying the recurrence relation (2.1), where $f_{n,k}$ is of the form $f_{n,k} = a_0 + a_1 k + a_2 n$.

3.2 Riordan matrix for $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ where $f_{n,k} = a_0+a_1k+a_2n$

Our approach proceeds step by step: we start with the simplest case, then apply the composition law.

3.2.1 The case $f_{n,k} = a_0 + a_1 k$

First, note that $F(n,0) = a_0^n$. Thus, by the definition, if the Riordan matrix $< g, f >$ exists, we have

$$g(z) = \sum_{n\in\mathbb{N}} \mathrm{F}(n,0)\frac{z^n}{n!} = \sum_{n\in\mathbb{N}} a_0^n \frac{z^n}{n!} = e^{a_0 z}, \text{ and}$$
$$f(z) = \left(k! e^{-a_0 z}\sum_{n\in\mathbb{N}} F(n,k)\frac{z^n}{n!}\right)^{\frac{1}{k}}.$$

However, Remark 2.6.1 guarantees that

$$\begin{aligned}\sum_{n\in\mathbb{N}} F(n,k)\frac{z^n}{n!} =& \frac{1}{k!a_1^k}\sum_{n\in\mathbb{N}}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(a_1j+a_0)^n\frac{z^n}{n!}\\ =& \frac{1}{k!a_1^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\Big(\sum_{n\in\mathbb{N}}(a_1j+a_0)^n\frac{z^n}{n!}\Big)\\ =& \frac{1}{k!a_1^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{(a_1j+a_0)z}\\ =& \frac{e^{a_0z}}{k!a_1^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{a_1jz}\\ =& \frac{e^{a_0z}}{k!a_1^k}(e^{a_1z}-1)^k.\end{aligned}$$

Hence we obtain the following.

Lemme 3.2.1. *In this case, the Riordan matrix corresponding to the sequence* $(F(n,k))_{n,k\in\mathbb{N}}$ *is given by*

$$(F(n,k))_{n,k\in\mathbb{N}} = \mathbb{F} =< e^{a_0z}, \frac{1}{a_1}(e^{a_1z}-1) > . \tag{3.4}$$

This formula also covers the case $a_1 = 0$ by taking the limit. Indeed, the limit of (3.4) as a_1 tends to 0 is $< e^{a_0z}, z >$, and

$$\begin{aligned}e^{a_0z}\frac{z^k}{k!} &= \sum_{n\in\mathbb{N}} a_0^n\frac{z^n}{n!}\frac{z^k}{k!}\\ &= \sum_{n\geq k} a_0^{n-k}\frac{n!}{k!(n-k)!}\frac{z^n}{n!} = \sum_{n\geq k}\binom{n}{k}a_0^{n-k}\frac{z^n}{n!}.\end{aligned} \tag{3.5}$$

From relation (1.10) in the case 1.3.2 and equation (3.5), we see precisely that $< e^{a_0z}, z >$ is the Riordan matrix associated with $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ when $a_1 = 0$.

3.2.2 The case $f_{n,k} = a_0 + a_2n$

To distinguish this from the previous subsection (at the level of notation only), here $f_{n,k}$ is $g_{n,k} = a_0 + a_2n$; the corresponding sequence is $(G(n,k))_{n,k\in\mathbb{N}}$ and the matrix is $\mathbb{G}$.

Proposition 3.2.1. *In this case, the Riordan matrix corresponding to the sequence* $(G(n,k))_{n,k\in\mathbb{N}}$ *is given by*

$$(G(n,k))_{n,k\in\mathbb{N}} = \mathbb{G} =< (1-a_2z)^{-\frac{a_0}{a_2}-1}, -\frac{1}{a_2}\ln(1-a_2z) > . \tag{3.6}$$

Proof. Remark 2.3.3 ensures that $\mathbb{G}$ should be the inverse of one of the matrices in the previous subsection, namely the one corresponding to $f'_{n,k} = -a_0 - a_2 - a_2k$, which is given by

$$(F'(n,k))_{n,k\in\mathbb{N}} = \mathbb{F}' =< e^{(-a_0-a_2)z}, -\frac{1}{a_2}(e^{-a_2z}-1) > .$$

Then we apply Proposition 3.1.1 to find the inverse of $\mathbb{F}'$. $\square$

This formula also includes the case $a_2 = 0$ by taking the limit. Indeed, the limit of $<(1-a_2z)^{-\frac{a_0}{a_2}-1}, -\frac{1}{a_2}\ln(1-a_2z)>$ as a_2 tends to 0 is $<e^{a_0z}, z>$. Using the equality (3.5), we then obtain the desired result.

Remarque 3.2.1. In the case $a_0 = 0$, this matrix becomes

$$(G_0(n,k))_{n,k\in\mathbb{N}} = \mathbb{G}_0 =< \frac{1}{1-a_2z}, -\frac{1}{a_2}\ln(1-a_2z) > .$$

3.2.3 The general case $f_{n,k} = a_0 + a_1k + a_2n$

To distinguish this from the previous subsections (again, only at the level of notation), here $f_{n,k}$ is $t_{n,k} = a_0 + a_1k + a_2n$; the corresponding sequence is $(T(n,k))_{n,k\in\mathbb{N}}$ and the matrix is $\mathbb{T}$. We state the result only for the case $a_2 \neq 0$ and $a_1 \neq 0$.

Théorème 3.2.1. *In this case, the Riordan matrix corresponding to the sequence $(T(n,k))_{n,k\in\mathbb{N}}$ is given by*

$$(T(n,k))_{n,k\in\mathbb{N}} = \mathbb{T} =< (1-a_2z)^{-\frac{a_0}{a_2}-1}, \frac{1}{a_1}\left((1-a_2z)^{-\frac{a_1}{a_2}} - 1\right) > . \tag{3.7}$$

Proof. Proposition 2.3.1 ensures that $\mathbb{T} = \mathbb{G}_0\mathbb{F}$. Hence

$$(T(n,k))_{n,k\in\mathbb{N}} = \mathbb{T} =< \frac{1}{1-a_2z}, -\frac{1}{a_2}\ln(1-a_2z) > \star < e^{a_0z}, \frac{1}{a_1}(e^{a_1z}-1) > .$$

Using the definition of the group law, we obtain the equation (3.7). □

Remarque 3.2.2. The limit of (3.7) as a_2 or a_1 tends to 0 yields (3.4) or (3.6), respectively.

3.3 Immediate consequence of these results

Using the definition of a Riordan matrix, we obtain:

Corollaire 3.3.1. *If $(F(n,k))_{n,k\in\mathbb{N}}$ satisfies the recurrence relation* (2.1) *for $f_{n,k} = a_2n + a_1k + a_0$, with $a_1 \neq 0$ and $a_2 \neq 0$, then*

$$\sum_{n\in\mathbb{N}} F(n,k)\frac{z^n}{n!} = \frac{1}{k!}(1-a_2z)^{-\frac{a_0}{a_2}-1}\left(\frac{1-(1-a_2z)^{-\frac{a_1}{a_2}}}{a_1}\right)^k.$$

Note that, by taking limits, this equation is consistent with the cases $a_1 = 0$ or $a_2 = 0$. Indeed,

$$\begin{aligned}
\sum_{n\in\mathbb{N}} F(n,k)\frac{z^n}{n!} &= \frac{1}{a_1^k k!}(1-a_2z)^{-\frac{a_0}{a_2}-1}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(1-a_2z)^{-\frac{a_1i}{a_2}} \\
&= \frac{1}{a_1^k k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i(1-a_2z)^{-\frac{a_0+a_1i+a_2}{a_2}} \\
&= \frac{1}{a_1^k k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\sum_{n\in\mathbb{N}}\binom{-\frac{a_0+a_1i+a_2}{a_2}}{n}(-1)^n a_2^n z^n \\
&= \sum_{n\in\mathbb{N}}\left(\frac{a_2^n n!}{a_1^k k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{n+i}\binom{-\frac{a_0+a_1i+a_2}{a_2}}{n}\right)\frac{z^n}{n!}.
\end{aligned}$$

Hence,

$$F(n,k) = \frac{a_2^n n!}{a_1^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{n+i} \binom{-\frac{a_0+a_1 i+a_2}{a_2}}{n}. \tag{3.8}$$

Then we use the classical identity for binomial coefficients with a negative upper argument:

$$\binom{-x}{n} = (-1)^n \binom{x+n-1}{n} = \frac{(-1)^n}{n!} (x)^{(\overline{n})}.$$

By setting $x = \dfrac{a_0 + a_1 i + a_2}{a_2}$, we obtain

$$\binom{-\frac{a_0+a_1 i+a_2}{a_2}}{n} = \frac{(-1)^n}{n!} \left(\frac{a_0 + a_1 i + a_2}{a_2}\right)^{(\overline{n})} = \frac{(-1)^n}{n!} a_2^{-n} \prod_{r=1}^{n} (a_0 + a_1 i + r a_2).$$

Substituting in the initial sum, the factors $a_2^n n!$ cancel out exactly:

$$a_2^n n! \, (-1)^{n+i} \binom{-\frac{a_0+a_1 i+a_2}{a_2}}{n} = (-1)^i \prod_{r=1}^{n} (a_0 + a_1 i + r a_2).$$

Thus, the whole expression simplifies to

$$F(n,k) = \frac{1}{a_1^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \prod_{r=1}^{n} (a_0 + a_1 i + r a_2). \tag{3.9}$$

Proposition 3.3.1. *Let $(f_{n,k})_{n,k\in\mathbb{N}}$, $(g_{n,k})_{n,k\in\mathbb{N}}$ and $(h_{n,k})_{n,k\in\mathbb{N}}$ be three sequences defined by*

$$f_{n,k} = a_0 + a_1 k + a_2 n, \qquad g_{n,k} = b_0 + b_1 k + b_2 n, \qquad h_{n,k} = (a_0 + b_0 + b_2) + b_1 k + a_2 n.$$

Let the associated matrices be $\mathbb{F} = (F(n,k))_{n,k\in\mathbb{N}}$, $\mathbb{G} = (G(n,k))_{n,k\in\mathbb{N}}$ and $\mathbb{H} = (H(n,k))_{n,k\in\mathbb{N}}$, defined by

$$\begin{array}{lcl} \mathrm{F}(0,0) = 1 & \textit{and} & F(n,k) = \mathrm{F}(n-1,k-1) + f_{n,k}\mathrm{F}(n-1,k), \\ \mathrm{G}(0,0) = 1 & \textit{and} & G(n,k) = \mathrm{G}(n-1,k-1) + g_{n,k}\mathrm{G}(n-1,k); \\ \mathrm{H}(0,0) = 1 & \textit{and} & H(n,k) = \mathrm{H}(n-1,k-1) + h_{n,k}\mathrm{G}(n-1,k). \end{array}$$

Then:

- *If $b_0 = a_1 - a_0 - a_2$, $b_1 = -a_2$ and $b_2 = -a_1$, then $\mathbb{G} = \mathbb{F}^{-1}$.*
- *If $b_2 = -a_1$, then $\mathbb{F}\mathbb{G} = \mathbb{H}$.*

These are consequences of the computation of the inverse of a Riordan matrix and the composition law, together with Proposition 3.1.2.

Proof. Let (g_a, f_a) and (g_b, f_b) be the Riordan representatives of $\mathbb{F}$ and $\mathbb{G}$ respectively. They are given by Theorem 3.2.1. Now set

$$< g, f > = < g_a, f_a > \ \star \ < g_b, f_b > .$$

(1) For the f-component:

$$f_a(z) = \frac{(1-a_2z)^{-\frac{a_1}{a_2}}-1}{a_1} \quad\Longrightarrow\quad 1+a_1f_a(z) = (1-a_2z)^{-\frac{a_1}{a_2}}.$$

$$\Longrightarrow\ f_b\circ f_a(z) = \frac{\Big(1-b_2f_a(z)\Big)^{-\frac{b_1}{b_2}}-1}{b_1} = \frac{\Big(1+a_1f_a(z)\Big)^{\frac{b_1}{a_1}}-1}{b_1} = \frac{(1-a_2z)^{-\frac{b_1}{a_2}}-1}{b_1}.$$

Thus, for any $c_0 \in \mathbb{R}$, if $c_2 = a_2$ and $c_1 = b_1$, then $f = f_c$.

(2) For the g-component:

$$g_a(z) = (1-a_2z)^{-\frac{a_0}{a_2}-1} = (1-a_2z)^{-\frac{a_0+a_2}{a_2}}, \qquad g_b(z) = (1-b_2z)^{-\frac{b_0+b_2}{b_2}}.$$

By composition,

$$g_b\circ f_a(z) = \Big(1-b_2f_a(z)\Big)^{-\frac{b_0+b_2}{b_2}} = \Big(1+a_1f_a(z)\Big)^{-\frac{b_0+b_2}{b_2}} = \Big((1-a_2z)^{-\frac{a_1}{a_2}}\Big)^{-\frac{b_0+b_2}{b_2}}.$$

With $b_2 = -a_1$,

$$g_b\circ f_a(z) = (1-a_2z)^{\frac{a_1}{a_2}\cdot\frac{b_0+b_2}{b_2}} = (1-a_2z)^{\frac{a_1}{a_2}\cdot\frac{b_0-a_1}{-a_1}} = (1-a_2z)^{\frac{a_1-b_0}{a_2}}.$$

Hence

$$g_a(z)\cdot(g_b\circ f_a(z)) = (1-a_2z)^{-\frac{a_0+a_2}{a_2}}\cdot(1-a_2z)^{\frac{a_1-b_0}{a_2}} = (1-a_2z)^{-\frac{a_0+b_0+a_2-a_1}{a_2}}.$$

Thus, if $c_0 = a_0+b_0-a_1 = a_0+b_0+b_2$, we have $g = g_c$. This gives the identifications $c_2 = a_2,\ c_1 = b_1,\ c_0 = a_0+b_0-a_1$. $\square$

This proposition shows that the inverse of the matrix associated with a sequence of the form (2.1) is again associated with a sequence of the same form; moreover the product is still of this form under suitable conditions.

Remarque 3.3.1. As a consequence of this proposition, we obtain a stronger result than in Proposition 2.3.1: even if the condition (2.12) does not hold but we only have $b_2 = -a_1$, we still have $\mathbb{X}\mathbb{Y} = \Big((Z(n,k))\Big)_{n,k\in\mathbb{N}}$ with $Z(n,k) = ||\mathcal{C}[z](n,k)||$, where $z_{n,k} = (a_0+b_0+b_2)+b_1k+a_2n$ if $x_{n,k} = a_0+a_1k+a_2k$ and $y_{n,k} = b_0+b_1k+b_2n$.

Remarque 3.3.2. We can also use this proposition to conclude that if $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ is the product of $\Big(w_{m,r}(n,k)\Big)_{n,k\in\mathbb{N}}$ with $\Big(W_{m,r}(n,k)\Big)_{n,k\in\mathbb{N}}$, then it satisfies

$$F(n,k) = (r+m-r-m)F(n-1,k)+F(n-1,k-1).$$

Taking the usual initial conditions into account, we obtain $F(n,k) = \delta_{n,k}$, which reaffirms Corollary 2.3.2.

3.4 Examples

These matrices are known from Sprugnoli [36], but we can see them as particular cases of our previous results. We have

$$\begin{aligned} \mathbb{P} &= \Big(\binom{n}{k}\Big)_{n,k\in\mathbb{N}} =< e^z, z >, \\ s &= \Big(s(n,k)\Big)_{n,k\in\mathbb{N}} =< 1, -\ln(1-z) >, \\ \mathbb{S} &= \Big(S(n,k)\Big)_{n,k\in\mathbb{N}} =< 1, e^z - 1 > . \end{aligned}$$

Although the r-Whitney numbers already appear in Cheon [7], we can also recover them here as particular cases, using the relations (3.4), (3.6) and (3.7) studied above:

$$\begin{aligned} \mathbb{W}_1 &= \Big(w_{m,r}(n,k)\Big)_{n,k\in\mathbb{N}} =< (1-mz)^{-\frac{r}{m}}, -\frac{1}{m}\ln(1-mz) >, \\ \mathbb{W}_2 &= \Big(W_{m,r}(n,k)\Big)_{n,k\in\mathbb{N}} =< e^{rz}, \frac{1}{m}\Big(e^{mz}-1\Big) >, \\ \mathbb{W}_3 &= \Big(WL(n,k)\big)_{n,l\in\mathbb{N}} =< (1-mz)^{-\frac{2r}{m}}, \frac{z}{1-mz} > . \end{aligned}$$

Next, let us focus on the case $m = 1$, the r-Stirling numbers:

$$\begin{aligned} s'_r &= \Big(w_{1,r}(n,k)\Big)_{n,k\in\mathbb{N}} =< (1-z)^{-r}, -\ln(1-z) >, \\ \mathbb{S}'_r &= \Big(S_r(n+r,k+r)\Big)_{n,k\in\mathbb{N}} =< e^{rz}, \Big(e^z - 1\Big) > . \end{aligned}$$

For $m = 0$, the matrices $\Big(W_{m,r}(n,k)\Big)_{n,k\in\mathbb{N}}$ give the generalized Pascal numbers:

$$\mathbb{P}_r =< e^{rz}, z > .$$

Using the definition of a Riordan matrix or directly Case 1.3.2 with $a = r$ and $b = 1$, we obtain

$$P_r(n,k) = r^{n-k}\binom{n}{k}. \tag{3.10}$$

For $r = 1$, we recover the Whitney numbers and the Lah numbers:

$$\begin{aligned} \mathbb{W}' &= \Big(w_m(n,k)\Big)_{n,k\in\mathbb{N}} =< (1-mz)^{-\frac{1}{m}}, -\frac{1}{m}\ln(1-mz) >, \\ \mathbb{W} &= \Big(W_m(n,k)\Big)_{n,k\in\mathbb{N}} =< e^z, \frac{1}{m}\Big(e^{mz}-1\Big) >, \\ \mathbb{L} &= \Big(WL(n,k)\big)_{n,l\in\mathbb{N}} =< (1-mz)^{-\frac{2}{m}}, \frac{z}{1-mz} > . \end{aligned}$$

For $r = 0$, we obtain the generalized Stirling numbers:

$$\begin{aligned} s_m &= \Big(s_m(n,k)\Big)_{n,k\in\mathbb{N}} =< 1, -\frac{1}{m}\ln(1-mz) >, \\ \mathbb{S}_m &= \Big(S_m(n,k)\Big)_{n,k\in\mathbb{N}} =< 1, \frac{1}{m}\Big(e^{mz}-1\Big) > . \end{aligned}$$

The numbers $\Big(S_m(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfy a recurrence relation similar to that of the ordinary Stirling numbers of the second kind $\Big(S(n,k)\Big)_{n,k\in\mathbb{N}}$, except that km appears instead of k. In other words, the weight of each East step $(n-1,k)\to(n,k)$ becomes km instead of k for $\Big(S_m(n,k)\Big)_{n,k\in\mathbb{N}}$. Hence

$$S_m(n,k) = m^{n-k}S(n,k). \tag{3.11}$$

A similar reasoning for $s(n,k)$ and $s_m(n,k)$ gives

$$s_m(n,k) = m^{n-k}s(n,k). \tag{3.12}$$

3.5 Some expressions in terms of Stirling numbers

In this section, all the sequences we consider are taken with the initial conditions 1 for $n=k=0$ and 0 if $n<\max(k,0)$.

Proposition 3.5.1. *[Case of r-Whitney numbers of the second kind] If* $T(n,k) = \mathrm{T}(n-1,k-1)+(a_0+a_1k)\mathrm{T}(n-1,k)$, *then*

$$T(n,k) = \sum_{j\in\mathbb{N}}\binom{n}{j}a_0^{n-j}a_1^{j-k}S(j,k).$$

Proof. We start from its corresponding Riordan matrix (2.6.1):

$$\begin{aligned}(T(n,k))_{n,k\in\mathbb{N}} = \mathbb{T} &=< e^{a_0z}, \frac{1}{a_1}(e^{a_1z}-1) > \\ &=< e^{a_0z}, z > \star < 1, \frac{1}{a_1}(e^{a_1z}-1) > \quad \text{(by (3.3))} \\ &= \mathbb{P}_{a_0}\mathbb{S}_{a_1}.\end{aligned}$$

Formulas (3.10) and (3.11) yield the result. □

Proposition 3.5.2. *[Case of r-Whitney numbers of the first kind] If* $A(n,k) = \mathrm{A}(n-1,k-1)+(a_0+a_2n)\mathrm{A}(n-1,k)$, *then*

$$A(n,k) = \sum_{j=k}^{j=n}\binom{j}{k}a_2^{n-j}(a_0+a_2)^{j-k}s(n,j).$$

Proof. Proposition 2.3.1 tells us that $\mathbb{A} = s_{a_2}\mathbb{P}_{a_0+a_2}$. We conclude using (3.10) and (3.12). □

Corollaire 3.5.1. *If* $F(n,k) = \mathrm{F}(n-1,k-1)+(a_0+a_1k+a_2n)\mathrm{F}(n-1,k)$, *then*

$$F(n,k) = \sum_{k\le i\le j\le n}\binom{j}{i}a_1^{i-k}a_2^{i-j}(a_0+a_2)^{j-k}s(i,j)\,S(i,k).$$

Proof. Proposition 2.3.1 gives $\mathbb{F} = \mathbb{A}\mathbb{S}_{a_1} = s_{a_2}\mathbb{P}_{a_0+a_2}\mathbb{S}_{a_1}$. □

3.6 A few words on the case where $b_{i,j}$ is non-trivial

In this section, we return to the fully general recurrence (1), where $b_{n,k}$ is arbitrary (and not assumed to be 1). The results from Section 1.5 will be useful.

Lemme 3.6.1. *Let $A(n,k) = a_n \mathrm{A}(n-1,k) + \alpha_n \mathrm{A}(n-1,k-1)$ and $B(n,k) = b_k \mathrm{B}(n-1,k) + \beta_k \mathrm{B}(n-1,k-1)$, with respective matrices $\mathbb{A}$ and $\mathbb{B}$. If $\mathbb{C} = \mathbb{A}\mathbb{B} = \Big(C(n,k)\Big)_{n,k\in\mathbb{N}}$, then the entries of $\Big(C(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfy*

$$C(n,k) = \alpha_n \beta_k \mathrm{C}(n-1,k-1) + (a_n + \alpha_n b_k)\mathrm{C}(n-1,k).$$

The reader can check this easily by a direct computation.

Proposition 3.6.1. *If $T(n,k) = (a_0 + a_1 k + a_2 n)\mathrm{T}(n-1,k) + (b_0 + b_1 k)\mathrm{T}(n-1,k-1)$, then*

$$T(n,k) = (b_0 + b_1|b_1)^{(\overline{k})2} \sum_{j\in\mathbb{N}} W_{a_1, a_0+a_2}(j,k)\, a_2^{n-j}\, s(n,j).$$

Here the notation W_{a_1,a_0+a_2} is slightly abusive: it denotes a sequence that satisfies the same recurrence as the r-Whitney numbers even when m, r are arbitrary. The weighted-path method used to construct it was described in equation (2.16).

Proof. In Lemma 3.6.1, take $a_n = a_0 + a_2 n$, $\alpha_n = 1$, $b_k = a_1 k$ and $\beta_k = b_1 k + b_0$. Then $(T(n,k))_{n,k\in\mathbb{N}}$ is the product of $(A(n,k))_{n,k\in\mathbb{N}}$ from Proposition 3.5.2 with $(T'(n,k))_{n,k\in\mathbb{N}}$ from Remark 1.3.2. Hence

$$\begin{aligned} T(n,k) =& \sum_{l\in\mathbb{N}} \left(\sum_{j\in\mathbb{N}} \binom{j}{l} a_2^{n-j} (a_0 + a_2)^{j-l} s(n,j) \right) (b_0 + b_1|b_1)^{(\overline{k})} a_1^{l-k} S(l,k) \\ =& (b_0|b_1)^{(\overline{k})} \sum_{j\in\mathbb{N}} a_2^{n-j} s(n,j) \left(\sum_{l\in\mathbb{N}} \binom{j}{l} (a_0 + a_2)^{j-l} a_1^{l-k} S(l,k) \right). \end{aligned}$$

We then apply Proposition 3.5.1. ∎

Combining Proposition 3.6.1 with Proposition 3.5.1 yields another proof of Neuwirth's result [2]. This is the explicit expression associated with (1) when $b_2 = 0$:

$$T(n,k) = (b_0 + b_1|b_1)^{(\overline{k})} \sum_{i,j\in\mathbb{N}} \binom{j}{i} a_0^{j-i} a_1^{i-k} a_2^{n-j} s(n,j)\, S(i,k). \tag{3.13}$$

Théorème 3.6.1. *Let $(T(n,k))_{n,k\in\mathbb{N}}$ be a sequence defined by*

$$T(n,k) = (a_2 n + a_1 k + a_0)\, T(n-1,k) + (b_2 n + b_1 k + b_0)\, T(n-1,k-1).$$

The array $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ admits a Riordan matrix representation $< g, f >$ if and only if one of the following conditions holds:

[2] with $(b_0 + b_1|b_1)^{(\overline{k})} = \prod_{i=0}^{k-1}(b_0 + b_1 + i b_1)$.

(i) $b_0, b_1 \neq 0$*,* $b_2 = -b_1$*,* $a_2 = 0$ *and* $a_1 b_0 + a_0 b_2 = 0$*;*

(ii) $b_0, b_1 \neq 0$*,* $b_2 = -b_1$*,* $a_0 = a_1 = -a_2 \neq 0$*;*

(iii) $b_1 = b_2 = 0$*.*

In case (i),

$$\begin{cases} f(t) = b_0\, t, \\ g(t) = e^{a_0 t}. \end{cases}$$

In case (ii),

$$\begin{cases} f(t) = b_0\, t, \\ g(t) = (1 - a_2 t)^{-\frac{a_0 + a_2}{a_2}} = 1. \end{cases}$$

In case (iii),

$$f(t) = \begin{cases} \frac{b_0}{a_1}\Big(e^{a_1 t} - 1\Big) & (a_1 \neq 0, a_2 = 0), \\ b_0 t & (a_1 = a_2 = 0), \\ \frac{b_0}{a_1}\Big((1 - a_2 t)^{-\frac{a_1}{a_2}} - 1\Big) & (a_1, a_2 \neq 0), \\ -\frac{b_0}{a_2} \ln(1 - a_2 t) & (a_1 = 0, a_2 \neq 0), \end{cases}$$

$$g(t) = \begin{cases} (1 - a_2 t)^{-\frac{a_0 + a_2}{a_2}} & (a_2 \neq 0), \\ e^{a_0 t} & (a_2 = 0). \end{cases}$$

Proof. Set

$$S_k(t) := \sum_{n \geq 0} T(n,k)\, \frac{t^n}{n!}.$$

For every $k \geq 1$ we have:

$$\begin{aligned} S_k'(t) =& \sum_{n \geq 1} T(n,k) \frac{t^{n-1}}{(n-1)!} \\ =& \sum_{n \geq 1} \Big((a_2 n + a_1 k + a_0)\, T(n-1,k) + (b_2 n + b_1 k + b_0)\, T(n-1,k-1)\Big) \frac{t^{n-1}}{(n-1)!} \\ =& (a_1 k + a_0) \sum_{n \geq 0} T(n,k) \frac{t^n}{n!} + (b_1 k + b_0) \sum_{n \geq 0} T(n,k-1) \frac{t^n}{n!} \\ & + t \sum_{n \geq 1} a_2 (n-1) T(n-1,k) \frac{t^{n-2}}{(n-1)!} + a_2 \sum_{n \geq 0} T(n,k) \frac{t^n}{n!} + b_2 \sum_{n \geq 0} T(n,k-1) \frac{t^n}{n!} \\ =& (a_1 k + a_0 + a_2) S_k(t) + (b_1 k + b_0 + b_2) S_{k-1}(t) + a_2 t S_k'(t) + b_2 t S_{k-1}'(t). \end{aligned}$$

Therefore,

$$(1 - a_2 t) S_k'(t) = (a_1 k + a_0 + a_2) S_k(t) + (b_1 k + b_0 + b_2) S_{k-1}(t) + b_2 t S_{k-1}'(t). \tag{3.14}$$

On the other hand, saying that $< g, f >$ is a Riordan matrix associated with $\Big(T(n,k)\Big)$ means that

$$S_k(t) = g(t) \frac{(f(t))^k}{k!},$$

which implies

$$\begin{aligned} S_k'(t) &= g'(t)\frac{(f(t))^k}{k!} + g(t)f'(t)\frac{(f(t))^{k-1}}{(k-1)!}, \\ \implies S_{k-1}(t) &= g(t)\frac{(f(t))^{k-1}}{(k-1)!}. \end{aligned}$$

We deduce

$$\frac{S_{k-1}(t)}{S_k(t)} = \frac{k}{f(t)}, \qquad \frac{S_k'(t)}{S_k(t)} = \frac{g'(t)}{g(t)} + k\frac{f'(t)}{f(t)}, \qquad \frac{S_{k-1}'(t)}{S_k(t)} = \left(\frac{g'(t)}{g(t)f(t)} - \frac{f'(t)}{f^2(t)}\right)k + \frac{f'(t)}{f^2(t)}k^2. \tag{3.15}$$

Dividing (3.14) by $S_k(t)$ and inserting (3.15), we get

$$(1-a_2t)\left(\frac{g'}{g} + k\,\frac{f'}{f}\right) = (a_1k + a_0 + a_2) \ + \ (b_1k + b_0 + b_2)\,\frac{k}{f} \ + \ b_2t\left[\left(\frac{g'}{gf} - \frac{f'}{f^2}\right)k + \frac{f'}{f^2}\,k^2\right].$$

Matching coefficients of k^0, k^1, k^2, we obtain:

$$(1-a_2t)\,\frac{g'}{g} = a_0 + a_2 \implies \frac{g'}{g} = \frac{a_0+a_2}{1-a_2t}; \tag{3.16}$$

$$(1-a_2t)\,\frac{f'}{f} = a_1 + \frac{b_0+b_2}{f} \ + \ b_2t\left(\frac{g'}{gf} - \frac{f'}{f^2}\right); \tag{3.17}$$

$$0 = \frac{b_1}{f} + b_2t\,\frac{f'}{f^2} \quad \implies \quad b_1f + b_2t\,f' = 0. \tag{3.18}$$

Equation (3.16) implies

$$g(t) = \begin{cases} K(1-a_2t)^{-\frac{a_0+a_2}{a_2}} & (a_2 \neq 0), \\ Ke^{a_0t} & (a_2 = 0), \end{cases} \tag{3.19}$$

and the condition $T(0,0) = 1$ forces $K = 1$.

First case: assume ($b_1 \neq 0$ or $b_2 \neq 0$). The conditions $f(0) = 0$ and $f'(0) \neq 0$, together with (3.18), in fact force $b_1 \neq 0$ and $b_2 \neq 0$. Solving (3.18) we obtain $f(t) = ct^{-\frac{b_1}{b_2}}$ for some constant c. The conditions $f(0) = 0$ and $f'(0) \neq 0$ then force

$$b_1 = -b_2, \qquad f(t) = ct, \quad c \neq 0. \tag{3.20}$$

Inserting (3.16) and (3.20) into (3.17) yields

$$\frac{1-a_2t}{t} = a_1 + \frac{b_0}{ct} + \frac{b_2(a_0+a_2)}{c(1-a_2t)} \implies 1 - a_2t = a_1t + \frac{b_0}{c} + \frac{b_2(a_0+a_2)t}{c(1-a_2t)}.$$

Multiplying by $c(1-a_2t)$ and matching coefficients of $1, t, t^2$, we obtain

$$0 = (b_0 - c) + \Big(ca_1 - b_0a_2 + b_2(a_0+a_2) + 2ca_2\Big)t - ca_2(a_1+a_2)t^2,$$

which implies

$$\begin{cases} b_0 = c \neq 0, \\ a_2(a_1 + a_2) = 0, \\ c(a_1 + a_2) + b_2(a_0 + a_2) = 0. \end{cases} \tag{3.21}$$

This leads precisely to cases (i) and (ii).

Second case: assume now that $b_1 = b_2 = 0$. Then (3.18) is trivial. Equation (3.17) becomes

$$(1 - a_2 t)\frac{f'}{f} = a_1 + \frac{b_0}{f}. \tag{3.22}$$

If $a_2 = 0$, we have $f' = a_1 f + b_0$. Using $f(0) = 0$ and $f'(0) \neq 0$, we obtain

$$f(t) = \begin{cases} \frac{b_0}{a_1}\Big(e^{a_1 t} - 1\Big) & (a_1 \neq 0), \\ b_0 t & (a_1 = 0). \end{cases}$$

If $a_2 \neq 0$, the solution to (3.22) with $f(0) = 0$ and $f'(0) \neq 0$ is

$$f(t) = \begin{cases} \frac{b_0}{a_1}\Big((1 - a_2 t)^{-\frac{a_1}{a_2}} - 1\Big) & (a_1 \neq 0), \\ -\frac{b_0}{a_2}\ln(1 - a_2 t) & (a_1 = 0). \end{cases}$$

This is exactly case (iii). □

Corollaire 3.6.1. *The array $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ admits a Riordan matrix representation if and only if it can be written in a form with $b_2 = b_1 = 0$.*

Proof. It is enough to show that cases (i) and (ii) are equivalent[3] to the situation in (iii). First, assume that $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfies (i). Then $a_0 \neq 0$[4] and

$$T(n,k) = \Big(-\tfrac{a_1 b_0}{a_0}n + \tfrac{a_1 b_0}{a_0}k + b_0\Big)T(n-1,k-1) + (a_1 k + a_0)T(n-1,k). \tag{3.23}$$

We will show that

$$F(n,k) = \binom{n}{k} a_0^{n-k} b_0^k.$$

satisfies the recurrence (3.23) with the same initial conditions. For $n = k$ this is clear. For

[3]in the sense that they generate the same sequence.

[4]We remain in the setting where $a_{n,k} \neq 0$.

$n > k$ we compute:

$$
\begin{aligned}
&\Big(-\tfrac{a_1b_0}{a_0}n+\tfrac{a_1b_0}{a_0}k+b_0\Big)F(n-1,k-1)+(a_1k+a_0)F(n-1,k)\\
=&\Big(-\tfrac{a_1b_0}{a_0}n+\tfrac{a_1b_0}{a_0}k+b_0\Big)\binom{n-1}{k-1}a_0^{n-k}b_0^{k-1}\\
&\quad+(a_1k+a_0)\binom{n-1}{k}a_0^{n-1-k}b_0^{k}\\
=&\binom{n-1}{k-1}a_0^{n-k}b_0^{k}+\binom{n-1}{k}a_0^{n-k}b_0^{k}\\
&\quad+a_1\Big((-n+k)\binom{n-1}{k-1}+k\binom{n-1}{k}\Big)a_0^{n-k-1}b_0^{k}\\
=&\binom{n}{k}a_0^{n-k}b_0^{k}\\
=&F(n,k).
\end{aligned}
$$

The expression $(-n+k)\binom{n-1}{k-1}+k\binom{n-1}{k}$ is zero: indeed, choosing k elements among $n-1$ and then pointing a distinguished element among the k chosen is equivalent to choosing $k-1$ elements first and then adding one distinguished element among the remaining $n-k$. We also observed in (1.10) that these sequences satisfy the recurrence

$$F(n,k)=a_0F(n-1,k-1)+b_0F(n-1,k),$$

which is a case of type $b_1=b_2=0$. We proceed similarly for (ii). The relation is

$$T(n,k)=(-b_1n+b_1k+b_0)T(n-1,k-1)+(a_2n-a_2k-a_2)T(n-1,k). \tag{3.24}$$

One can show that $\big(T(n,k)\big)_{n,k\in\mathbb{N}}$ also satisfies

$$T(n,k)=b_0T(n-1,k-1)+(a_2n-a_2k-a_2)T(n-1,k). \tag{3.25}$$

To see this, note that (3.24) and (3.25) are equivalent to

$$T(n,k)=\delta_{n,k}b_0^k. \tag{3.26}$$

For $n=k$ this is clear. For $k=n-1$, since $T(1,0)=0$ we get

$$\forall n\in\mathbb{N}:\ T(n,n-1)=0\implies\forall k<n-1:\ T(n,k)=0.$$

Thus the two relations (3.24) and (3.25) are equivalent, which again reduces the situation to case (iii). $\square$

Conclusion: Perspectives

In this thesis, we have studied three approaches to obtain the solutions of the array $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$. Although the methods are distinct, they often lead to similar results.

We have been able to write the Riordan matrices for all cases of the form

$$\mathrm{F}(0,0)=1 \qquad \text{and} \qquad F(n,k)=\mathrm{F}(n-1,k-1)+(a_2n+a_1k+a_0)\mathrm{F}(n-1,k),$$

and therefore to obtain explicit formulas for these sequences.

The grammatical method, combined with combinatorial and analytic differential equations, contains a great deal of information and many interpretations, but its manipulation requires prior knowledge of formal power series expansions and the resolution of differential equations. This method is very rich because it links a grammar à a combinatorial species, which, being a functor, carries a lot of structural information.

The approach based on weighted paths is interesting because it does not require many prerequisites to be understood. For the arrays $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ with $b_2=0$, this path-based approach shows that one can obtain identities quickly and in a simple way. For the other cases, we have given a general formula on which one can work in order to derive simpler relations.

The Riordan matrix approach consists in viewing the array $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ as a pure matrix built from formal power series. It is sometimes easy to handle when the pair $<g,f>$ representing $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ is known. Unfortunately, there are several cases that it cannot deal with.

Thus, each of these approaches is effective depending on the type of sequences to be studied and on the techniques with which we are familiar. As we have done in some instances, combining them may be a good idea to obtain more interesting results. This technique may require a better understanding of the algebra of these sequences, in particular inverse products or other operations that could be useful to express a more complex case in terms of simpler ones.

We also concluded by writing all the arrays $\Big(F(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfying (2.1) as a transition matrix, using weighted paths. It would be interesting to see whether one can write all the arrays $\Big(T(n,k)\Big)_{n,k\in\mathbb{N}}$ satisfying equation (1) in the form of a transition matrix. Some generalizations of Eulerian numbers have already been written in this way. We wonder whether one can recover these results and extend them to all cases of type (1).

An interesting question would be to obtain a simpler explicit formula for the case $b_2\neq 0$ using these approaches. Moreover, as a natural extension of this thesis, we would like to apply these methods to the study of more general recurrences. Some number sequences already satisfy recurrence relations of the form

$$t_{n,k}=(a_1n+a_2k+a_3)t_{n-1,k}+(b_1n+b_2k+b_3)t_{n-1,k-1}+(c_1n+c_2k+c_3)t_{n-1,k-2},$$

or even more general ones.

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