# Lie Groups and Lie Algebras: Curvatures and Representations

AUBERT Voalaza Mahavily Romuald (aubert@aims.ac.za)
International Centre for Theoretical Physics (ICTP)

Supervised by: Professor Claudio Arezzo and Shubham Sinha
Mathematics section

14 August 2025

*Submitted in partial fulfillment of a Diploma programme*

# Abstract

Lie groups form a natural framework for describing continuous symmetries in geometry, analysis, and mathematical physics. Their associated Lie algebras capture the infinitesimal structure of the group, and the exponential map links these two perspectives. This thesis investigates the interplay between the algebraic and geometric aspects of Lie groups, with a particular emphasis on left-invariant Riemannian metrics, Cartan connections, and representation theory.

After reviewing the necessary background on smooth manifolds, Riemannian geometry, and Lie theory, we study the Lie-theoretic and Riemannian exponential maps, comparing their properties and highlighting cases where they coincide or differ. Using Milnor's framework, we perform explicit curvature computations for Lie groups with left-invariant metrics. We also explore the representation theory of complex semisimple Lie algebras, presenting examples for $\mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(3,\mathbb{C})$, and more general cases.

The main goal is to study representations of Lie algebras, which can contain more information about Lie groups and are often easier to analyze than the groups themselves. In this thesis, we focus only on complex cases, but we expect to be able to address the real ones later using realification or complexification theory.

This project will allow me to learn about Lie groups, Lie algebras, their curvature, and their representations.

**Keywords**: Lie theory, representation theory, Lie groups, Lie algebras, differential geometry.

# Acknowledgements

I want to acknowledge God, who always extends His grace and reigns over my life. I am here now by His will and His promise. I cannot express how great He is or how good He has been in my life and studies, through the Lord and Savior, Jesus Christ. Glory belongs solely to His mighty name.

I would like to thank the ICTP leadership, particularly Prof. Lothar Göttsche, the Diploma Programme ICTP, Coordinator for Mathematics, for everything, especially the education they provided us at ICTP. I am really grateful to have been chosen to be part of this program. This is a great opportunity for me to achieve my goal.

I deeply thank my supervisors, Prof. Claudio Arezzo, and Sinha Shubham for their guidance, suggestions, and for giving me freedom during this project. Thank you for accepting me among your students and for your time for our frequent discussions.

I would like to thank all the lecturers, tutors, and all ICTP staff who shared their wonderful knowledge and experiences with us. I learned and gained many things during this program.

I am also grateful to my parents and family: both biological and spiritual, for their prayers and encouragement throughout my studies. "Misaotra indrindra" (Thank you very much). Their trust and encouragement are an extra motivation for me; their words like *"Tahin' i Tompo iha, koahy"* ("You are blessed by the Lord, my son") and *"Mandresy ianao zanako"* ("You will win, my child") have given me strength during times when life was sometimes hard. As we believe, *"tsodrano zavamahery"* ("a blessing is powerful").

I would like to thank the professors and staff in the *Faculté des Sciences, Université de Fianarantsoa* who provided me with the background that allowed me to work in mathematics and facilitated the procedures for the necessary documents required to go to AIMS and also to come here to Italy at ICTP.

Finally, I thank all of my ICTP Diploma Mathematics cohort, with whom I have learned, exchanged, and discussed many good mathematics. I have had the opportunity to meet many brilliant people.

# Contents

# Introduction

Lie groups and Lie algebras play a major role in modern mathematics, making them one of the most standard languages for describing continuous symmetries in geometry, analysis, and mathematical physics. They stem from the exploratory research of Sophus Lie, who in the late $19$-th century discovered that they provide an appropriate mathematical context for describing both global and infinitesimal symmetries. A Lie group is both a differentiable manifold and a group, with the property that the two structures are compatible such that both the multiplication and inversion operations are smooth maps. Instead of being defined only by its group operation, a Lie group can alternatively be described by its associated 'Lie algebra,' which captures the infinitesimal behaviour near the identity element. This picture of interplay between these two perspectives, is captured by the exponential map.

Beyond their intrinsic beauty, Lie groups and their representations play a decisive role in diverse areas of mathematics and physics. In differential geometry, they arise naturally as transformation groups of manifolds, with applications to the study of homogeneous spaces and symmetric spaces Helgason (2024). The theory of semisimple Lie algebras, in particular, provides a systematic way to study high-dimensional symmetry via the representation theory of finite-dimensional modules.

From a geometric perspective, left-invariant Riemannian metrics on Lie groups yield rich families of spaces whose curvature can often be computed explicitly, revealing deep structural properties Milnor (1976). This connection has been developed extensively in the context of symmetric spaces and Cartan geometries, where Lie groups act transitively by isometries, and their isotropy subgroups encode the local geometry. The advantage of having a bi-invariant metric is that geodesics can be understood using the Lie exponential map, which is not always true when only a left-invariant metric is present. Then the Cartan connection can change the role of the Levi-Civita connection so it still holds.

The motivation for this thesis is to understand Lie groups and their geometry. We aim to study Lie groups via their Lie algebras, and also through the representations of these Lie algebras, that can help as to understand some class of Lie groups.

The structure of the thesis is as follows. Chapter 1 reviews the necessary background on smooth and Riemannian manifolds. Chapter 2 introduces Lie groups, Lie algebras, and their representations. Chapter 3 focuses on the exponential map, its consequences, and its interaction with Riemannian geometry, including a discussion of Lie groups with left-invariant metrics and Cartan connections. Chapter 4 develops the representation theory of semisimple Lie algebras, with examples from $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(3,\mathbb{C})$ and more general complex semisimple Lie algebras. This goal applies to all Lie groups and Lie algebras, both real and complex, but here we start with the complex semisimple ones. Finally, Chapter 5, we end with some remarks.

# 1. On Smooth Manifolds and Riemannian Manifolds

In this section, we start with important facts about smooth manifolds that will be useful for the next part of this dissertation. It will be mostly about smooth manifolds, smooth maps submanifolds and Riemannian manifolds. The main references of this section are Lee (2003) and Do Carmo and Flaherty Francis (1992).

## 1.1 Smooth manifolds and smooth maps

**1.1.1 Definition** (Smooth manifold)**.** A smooth manifold $M$ of dimension $n$, consists of

(i) a topological space $M$, Hausdorff and second countable;

(ii) a smooth atlas which is collection of pairs $(U_i, \phi_i)_{i\in I}$, $\phi_i : U_i \to \tilde{U}_i$;
such that:

- $\phi_i$ is a homeomorphism where $(\tilde{U}_i)_{i\in I}$ are open subsets of $\mathbb{R}^n$ (namely they are charts) and $(U_i)_{i\in I}$ is an open cover of $M$ ;
- for any $i, j \in I$: $\phi_j \circ \phi_i^{-1} : \phi_i[U_i \cap U_j] \to \phi_j[U_i \cap U_j]$ are smooth $\mathcal{C}^\infty$.

(iii) the atlas is maximal, in the sense that if $(U, \phi)$ and for any $i \in I, \phi \circ \phi_i^{-1}, \phi_i \circ \phi^{-1}$ are smooth on $\phi_i[U_i \cap U], \phi[U \cap U_i]$, then $(U, \phi)$ must belong to the atlas.

For such above notation, in the next, we describe a manifold $M = (M, (U_i, \phi_i, \tilde{U}_i)_{i\in I}, n)$ of dimension $n$. So, in the next of this dissertation, $M$ is always with these structure, unless we mention the opposite. In practice to check that a Hausdorff, second countable topological space is a manifold, it is enough to see that it is locally homeomorphic to $\mathbb{R}^n$ and the composition of the maps are smooth. Indeed, we can neglect the maximality of atlas because we can always construct an unique maximal one from a given atlas.

✎ **Notice.** One can define similarly the complex manifold like smooth manifold above, space locally homeomorphic to $\mathbb{C}^n$ and the transition maps required to be holomorphic. One can see directly is any complex manifold of dimension $m$ is smooth manifold of dimension $n = 2m$ and if a smooth manifold of dimension $n$ is also complex manifold, $n$ must be even.

**1.1.2 Lemma.** For any smooth atlas $(U_\alpha, \phi_\alpha)_{\alpha\in A}$,

$$\mathcal{A} = \{(U, \phi) \text{ chart suth that for any } \alpha \in A : \quad \phi_\alpha \circ \phi^{-1}, \phi \circ \phi_\alpha^{-1} \text{ are smooth}\} = (U_i, \phi_i)_{i\in I}$$

is the unique maximal atlas that contains $(U_\alpha, \phi_\alpha)_{\alpha\in A}$.

*Proof.* Clearly, $A \subset I$. To prove that it is an atlas, the only thing that we need to check is the case where both of $i, j$ is in $I \setminus A$. We want to show that $\phi_j \circ \phi_i^{-1}$ is locally smooth on $\phi_i[U_i \cap U_j]$, then smooth. Let $\tilde{p} \in \phi_i[U_i \cap U_j]$ and $p = \phi_i^{-1}(\tilde{p})$. There is $\alpha \in A$ that contains $p$. Then $p \in \phi_i[U_i \cap U_j \cap U_\alpha]$ and $\phi_j \circ \phi_i^{-1} = (\phi_j \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi_i^{-1})$ is smooth on $\phi_i[U_i \cap U_j \cap U_\alpha]$. It is maximal by definition as $A \subset I$. The uniqueness is by similar reasoning. Assume, $(U_j, \phi_j)_{j\in J}$ and $(U_k, \phi_k)_{k\in K}$ such that $A \subset J \cap K$. By symmetry, we will prove only $(U_k, \phi_k)_{k\in K} \subset (U_j, \phi_j)_{j\in J}$. Fix an arbitrary $k \in K$ and let's show that $\phi_k \circ \phi_j^{-1}$ and $\phi_j \circ \phi_k^{-1}$, so $(U_k, \phi_k) \in (U_j, \phi_j)_{j\in J}$. We pick again $\tilde{p} \in \phi_k^{-1}(U_k \cap U_j) \in U_\alpha$, for some $\alpha \in A$ to get that $\phi_j \circ \phi_k^{-1}$ is smooth near to $\tilde{p}$. □

**1.1.3 Remark.** Every open subset $U$ of a smooth manifold is trivially itself smooth manifold because we can just restrict each chart on its domain intersects $U$.

**1.1.4 Definition** (Smooth map)**.** Let $M$ and $N$ be two manifolds whose the smooth atlas are $(U_i, \phi_i)_{i\in I}$ and $(V_i, \varphi_i)_{i\in J}$. The map $F : M \to N$ is said to be smooth if

$$\text{for each } i \in I, j \in J, \text{ the map } \varphi_j \circ F \circ \phi_i^{-1} \text{ is smooth on } \phi_i(U_i \cap F^{-1}[V_j]).$$

Again, as we mostly work on charts which can cover the manifolds that is atlas simpler to work with but this definition requires as to check for all possible charts (maximal atlas). Therefore, this version is sometimes less practical. There is the next equivalent definition is important.

**1.1.5 Definition** (Smooth map)**.** With the assumptions in Definition 1.1.4, $F$ is smooth if

$$\text{for each } p \in M, \text{there are } (U, \phi), \text{ chart in } M,\ p \in U \text{ and } (V, \varphi), \text{ chart in } N,\ F(p) \in V \text{ such that } \varphi \circ F \circ \phi^{-1} \text{ is smooth on } \phi(U \cap F^{-1}[V]).$$

Clearly Definition 1.1.4 implies 1.1.5. Conversely if we have 1.1.5, let's try to prove 1.1.4. Let $i \in I, j \in J$. To show $\varphi_j \circ F \circ \phi_i^{-1}$ is smooth on $\phi_i(U_i \cap F^{-1}[V_j])$, recall, from calculus that being $\mathcal{C}^\infty$ for map between subset of $\mathbb{R}^n$ is a local property. Then, it is enough to prove that for each $\tilde{p} \in$, $\varphi_j \circ F \circ \phi_i^{-1}$ is $\mathcal{C}^\infty$ near to $\tilde{p}$. Consider $p = \phi_i^{-1}(\tilde{p})$ and apply 1.1.5 to get charts $(U, \phi), (V, \varphi)$ containing $p$ and $F(p)$ such that $\varphi \circ F \circ \phi^{-1}$ is smooth on $\phi(U \cap F^{-1}[V])$. From the structure of the smooth manifolds, we have $\phi \circ \phi_i^{-1}$ and $\varphi_j \circ \varphi^{-1}$ are respectively smooth on $\phi_i[U \cap U_i]$ and $\varphi[V_j \cap V]$. Thus $\varphi_j \circ F \circ \phi_i^{-1} = (\varphi_j \circ \varphi^{-1}) \circ (\varphi \circ F \circ \phi^{-1}) \circ (\phi \circ \phi_i^{-1})$ is smooth on $\phi_i[U \cap U_i \cap F^{-1}[V_j]]$, which is an open neighbourhood of $\tilde{p}$.
It is clear that the smoothness is a local property.

**1.1.6 Remark.** Smoothness implies continuity. It is because this is known for the case subsets of $\mathbb{R}^n$ and the fact the continuity is local property. Indeed, if $p \in M$, we choose the charts $(U_i, \phi_i)$ and $(V_j, \varphi_j)$ that contain $p$ and $F(p)$ with $\varphi_j \circ F \circ \phi_i^{-1}$ is smooth on $\phi_i(U_i \cap F^{-1}[V_j])$. Then $\varphi_j \circ F \circ \phi_i^{-1}$ is continuous on $\phi_i(U_i \cap F^{-1}[V_j])$ and $F = \varphi_j \circ (\varphi_j \circ F \circ \phi_i^{-1}) \circ \phi_i$ is continuous on $U_i \cap F^{-1}[V_j]$, which is open neighbourhood of $p$.

**1.1.7 Remark** (On product)**.** It is straightforward to see that for any smooth manifolds $M$ and $N$, $M \times N$ is also smooth manifolds. The charts are just the products. Now, we have the following results.

(i) If we have anothers smooth manifolds $M', N'$ with $f_1 : M \to M'$ and $f_2 : N \to N'$ are both smooth. The product map $f_1 \times f_2 : M \times N \to M' \times N'$, $(m, n) \mapsto (f(m), g(n))$ is also smooth. If $(p, q) \in M \times N$, there are $(U_i, \phi_i)$,$(V_j, \varphi_j)$, $(W_k, \rho_k)$ and $(O_l, \delta_l)$ respectively charts in $M, M', N$ and $N'$, that contain respectively $p, f_1(p), q$ and $f_2(p)$ such that $\varphi_j \circ f_1 \circ \phi_i^{-1}$ and $\delta_l \circ f_2 \circ \rho_k^{-1}$ are smooth. Hence $(\varphi_j \times \delta_l) \circ (f_1 \times f_2) \circ (\phi_i^{-1} \times \rho_k^{-1}) = (\varphi_j \circ f_1 \circ \phi_i^{-1}) \times (\delta_l \circ f_2 \circ \rho_k^{-1})$ is smooth.

(ii) We have also a diffeomorphism[1] $M \times N \to N \times M, (x, y) \mapsto (y, x)$.

(iii) The projections are smooth. Again, the continuity is known in the level of topology. For $(p, q) \in M \times N$, there are $(U_i, \phi_i)$ and $(V_j, \varphi_j)$ charts in $M$ and $N$ containing $p$ and $q$. $\phi_i \circ \pi_M \circ (\phi_i \times \varphi_j)^{-1}$ and $\varphi_j \circ \pi_M \circ (\phi_i \times \varphi_j)^{-1}$ are just projections which are trivially smooth.

(iv) For any smooth maps $f : X \longrightarrow M$ and $g : X \longrightarrow N$, $f \times g : X \longrightarrow M \times N$, $x \mapsto (f(x), g(x))$ is also smooth. Hence the universal property.

(v) Fixing $p \in M$, the inclusion map $i_p : N \to M \times N$ defined by $i_p(y) = (p, y)$ is smooth. Let

[1] smooth homeomorphism, whose inverse also smooth.

$y \in N$ and consider the charts $(U_i, \phi_i)$ and $(V_j, \varphi_j)$ in $M$ and in $N$, which contains $p$ and $y$. Then $(U_i \times V_j, \phi_i \times \varphi_j)$ is a chart that contains $i_p(x)$ and $(\phi_i \times \varphi_j) \circ i_p \circ \varphi_j^{-1}$ is the inclusion $z \mapsto (\phi_i(p), z)$ which is trivially smooth. The same applies if we fix $q \in N$ and $j_q(x) = (x, q)$.

As in $\mathbb{R}^n$, we know that the differentiability is a local property and the restriction of differentiable map is also differentiable. These properties still work in general smooth manifold.

**1.1.8 Proposition.** With the assumptions in Definition 1.1.4, we have.

(a) If $F$ is smooth then for any open subset $U$ of $M$, the restriction $F|_U : U \to N$ is smooth.

(b) If any point of $M$ admits open neighbourhood $U$ which $F|_U$ is smooth, then, $F$ is smooth

*Proof.* (a) is straightforward by the second definition, as restriction of differentiable map is differentiable in $\mathbb{R}^n$. For (b), pick $p \in M$; by assumption $p \in U$ and $F|_U : U \to N$ is smooth. Then there is $k \in I, j \in J$ such that $p \in U_k$, $F(p) \in V_j$ and $\varphi_j \circ F|_U \circ (\phi_k|_U)^{-1} = \varphi_j \circ F \circ (\phi_k|_U)^{-1}$ is smooth. So, it is enough to prove that $(U \cap U_k, \phi_k|_U) \in (U_i, \phi_i)_{i \in I}$. To see it, we just need to use the fact $\phi_i[U_k \cap U_i \cap U]$ is open subset of $\phi_i[U_i \cap U]$ and again restriction of differentiable map is differentiable in $\mathbb{R}^n$. The same for $\phi_i[U_k \cap U_i \cap U] \subset \phi_i[U_i \cap U]$, so, we get $\phi_i \circ (\phi_k|_U)^{-1}$ and $(\phi_k|_U) \circ \phi_i^{-1}$. □

We have defined objects (the collections of smooth manifolds) and morphisms (the smooth map), we expect to get a category. So we want to check if the composition is well-defined.

**1.1.9 Remark.** The good news is the composition of smooth maps is smooth. It is straightforward. Indeed, consider $M \xrightarrow{F} N \xrightarrow{G} P$ be two smooth maps (denote $(W_k, \delta_k)_{k \in K}$ atlas on $P$). Let $i \in I$ and $k \in K$. To prove $\delta_k \circ G \circ F \circ \phi_i^{-1}$ is smooth on $\phi_i[U_i \cap F^{-1} \circ G^{-1}[W_k]]$, let's do it locally. Pick $\tilde{p} \in \phi_i[U_i \cap F^{-1} \circ G^{-1}[W_k]]$ so $p = \phi_i(\tilde{p}) \in U_i \cap F^{-1} \circ G^{-1}[W_k]$ and $F(p) \in G^{-1}[W_k]$. Then there exists, $(V_j, \varphi_j)$ chart in $N$ that contains $F(p)$ such that $\delta_k \circ G \circ \varphi_j^{-1}$ and $\varphi_j \circ F \circ \phi_i^{-1}$ are smooth. Then $\delta_k \circ G \circ F \circ \phi_i^{-1} = (\delta_k \circ G \circ \varphi_j^{-1}) \circ (\varphi_j \circ F \circ \phi_i^{-1})$ is smooth on $\phi_i[U_i \cap F^{-1} \circ G^{-1}[W_k] \cap F^{-1}[V_j]] \ni \tilde{p}$.

Then, smooth manifolds form a category. The isomorphisms are the the diffeomorphisms.

## 1.2 Derivations, Vector fields, Tangent spaces, Differential maps

Let $A$ be an algebra over a field $\mathbb{K}$. A derivation $D$ on $A$ is its linear endomorphism satisfying the product rules, that is $D(fg) = fD(g) + D(f)g$. The algebra with which we shall mainly deal is the algebra of smooth functions on a smooth manifold $M$, denoted by $\mathcal{C}^\infty(M)$.

**1.2.1 Definition.** For a smooth manifold $(M, n)$, we mean by a vector field on $M$ is just a derivation on $\mathcal{C}^\infty(M)$. A derivation at $p$ is just $X_p : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}, f \mapsto X(f)(p)$ for some $X$ vector field on $M$.

We will denote $\mathfrak{X}(M)$ the set of vector fields on $M$. The set $T_pM := \{X_p : X \text{ vector field on } M\}$ is a vector space of dimension $n$.

**1.2.2 Definition.** For a smooth map $F : M \longrightarrow N$, the differential of $F$ at a point $p \in M$ is a linear map

$$\begin{aligned} (\mathrm{d}F)_a \colon T_pM &\longrightarrow T_{F(p)}N \\ X_p &\longmapsto (\mathrm{d}F)_a(X_p) = Y_{F(p)}, \end{aligned}$$

where $Y$ is vector field such that $Y_{F(p)}(g) = X_p(g \circ F)$[2] for any $g \in \mathcal{C}^\infty(N)$.

[2]Such vector field $Y$ exists.

Another view of tangent space $T_p$ is set of vector $v$ where there is smooth curve $\alpha$ on $M$, defined around $p$, where $\alpha'(0) = v$ and $\alpha(0) = p$. In this point of view,

$$(\mathrm{d}F)_p(\alpha'(0)) = \frac{d}{dt}\Big|_{t=0} F \circ \alpha(t).$$

## 1.3 Submanifolds: embedded vs immersed

At the topological level, any subset $X$ of a topological space $M$ inherits a topology that makes the inclusion a topological embedding. This topology is natural in $M$.
Now, when a smooth structure is added, we require more conditions to be able to get a smooth manifold structure on $X$, so that it is nice, in the sense that we will try to define a submanifold.
Intuitively, we expect that any chart $(U_i, \phi_i)$ of $M$ should induce a chart on $X$, namely that $(X_i = U_i \cap X, \phi_i|_{X_i})$ should produce a smooth atlas for $X$. But, in the end, it is not exactly that we need; rather, we will reformulate it in the sense that we can put a smooth structure on $X$ that is induced (in a natural way) from the structure on $M$. However, not every subset of $M$ satisfies such a condition.

**1.3.1 Definition** (Embedded Submanifold)**.** Let $(M, (U_i, \phi_i, \tilde{U}_i)_{i\in I}, n)$ is manifold. $X \subset M$ is said a submanifold of $M$, of dimension $k \in \mathbb{N}$, $k \leq n$ if $X$ can be covered by $(U_i)_{i\in A}$ $(A \subset I)$ such that

$$\forall i \in A : \quad \phi_i(X \cap U_i) = \mathbb{R}_k \cap \tilde{U}_i, \tag{1.3.1}$$

where $\mathbb{R}_k = \{x \in \mathbb{R}^n : x_j = 0 \text{ for } j > k\}$ homeomorphic to $\mathbb{R}^k$.

**1.3.2 Definition** (Smooth immersion)**.** A smooth map $M \xrightarrow{F} N$ is a smooth immersion if for each $p \in M$, $(dF)_p$ is injective. Moreover, if $F$ is topological embedding, we say $F$ is smooth embedding. Dually, $F$ is smooth submersion if $(dF)_p$ is surjective for each $p \in M$.

**1.3.3 Lemma.** Under the assumption in Definition 1.3.1 and (1.3.1), let $\mathbb{R}_k \xrightarrow{\varepsilon_k} \mathbb{R}^k$ be the natural homeomorphism. Let $X_i = U_i \cap X$ and $\rho_i = \varepsilon_k \circ \phi_i|_{X_i}$. We have $(X_i, \rho_i)_{i\in A}$ is smooth atlas on $X$ and the inclusion is a smooth immersion.

From now on, if we think $X$ as submanifold of $M$, with the topology induced, $X$ is also smooth manifold with the atlas constructed in Lemma 1.3.3.

**1.3.4 Proposition.** Let $M$ and $X$ be two smooth manifolds such that $X \subset M$ (the topology on $X$ is the induced one from $M$). If the inclusion map is a smooth immersion then $X$ is submanifold of $M$. and the smooth structure on $X$ is the same than we constructed in Lemma 1.3.3.

**1.3.5 Example.** Let $M \xrightarrow{F} N$ be an embedding. We expect $F(M)$ to be identifiable to $M$. Clearly, they can be diffeomorphic as they are homeomorphic and we can push forward the charts in $M$ via $F$. In this case, the inclusion $F(M) \hookrightarrow N$ becomes a composition of diffeomorphism and embedding then embedding. Hence $F(M)$ is an (embedded) submanifold of $N$.

Now, how about if we have only smooth immersion, can we still have $F(M)$ is submanifold? First to be able (easy) to identify $F(M)$ with $M$, via $F$, we require $F$ to be injective. Even though issues arise at the topological level, such cases are still interesting, which is called immersed submanifold. Namely, we can change the topology in $F(M)$ (not necessarily the induced from $N$), with specific smooth structure to get the inclusion $F(M) \hookrightarrow N$ becomes a smooth immersion. The topology and smooth structure are both the push forward from $M$ via $F$ that make $F$ diffeomorphic onto $F(M)$.

**1.3.6 Definition** (Immersed Submanifold)**.** Let $M$ and $X$ be be two smooth manifolds such that $X \subset M$ (the topology on $X$ is not necessarily the one induced from $M$). If the inclusion map is a smooth immersion, we say that $X$ is an immersed submanifold of $M$.

Some example will be seen in Lie subgroups section, for Example 2.1.8.

The following propositions are the combination of some theorems or propositions in Lee (2003).

**1.3.7 Proposition.** Let $M \xrightarrow{F} N$ be a smooth map.

(a) For $p \in M$, if $(dF)_p$ is injective (resp surjective) then $F$ is a smooth immersion (resp submersion) near to $p$. Also, if $(dF)_p$ is isomorphism, $F$ is diffeomorphism near to $p$.

(b) If $F$ is bijective and local diffeomorphism then it is diffeomorphism. Also smooth immersion and smooth submersion implies local diffeomorphism.

**1.3.8 Proposition.** Let $(M, (U_i, \phi_i, \tilde{U}_i)_{i \in I}, m) \xrightarrow{F} (N, (V_i, \varphi_i, \tilde{V}_i)_{i \in I}, n)$ be smooth map with constant rank $r$, that is

$$\forall p, q \in M : \quad rank((dF)_p) = rank((dF)_q) = r.$$

Then the followings hold.

(a) For each $p \in M$ there is $i \in I$ and $j \in J$ such that $p \in U_i, F(p) \in V_j$ and

$$\hat{F}(x_1, \cdots, x_m) := \varphi_j \circ F \circ \phi_i((x_1, \cdots, x_m) = (x_1, \cdots, x_r, 0, \cdots 0).$$

(b) $F$ is smooth immersion if it is injective. Also it smooth submersion if it is surjective.

**1.3.9 Remark.** Item (a) is a necessary and sufficient condition for a smooth map to have constant rank. More generally, it is enough that $\hat{F}$ is linear.

## 1.4 Quotient Manifolds

This part is almost from Lee (2003). The goal of this section is to note that not every quotient topology naturally gives a manifold structure and to state some important facts in which case this quotient also has a smooth structure. First, a continuous map $f : X \longrightarrow Y$ is called proper if the preimage of a compact subset of $Y$ is compact in $X$. For a group action, if $G$ acts on a manifold $M$, we denote $M/G$ the set of orbits with the quotient topology, called the orbit space. Assume now that $G$ is a topological group. The action is proper if it is continuous and the map $G \times M \to M \times M$, $(g, p) \mapsto (g \cdot p, p)$ is a proper map.

**1.4.1 Theorem** (Quotient Manifold Theorem)**.** *Suppose $G$ is a Lie group acting smoothly, freely, and properly on a smooth manifold $M$. Then the orbit space $M/G$ is a topological manifold of dimension equal to $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

The proof can be found in Lee (2003), together with many interesting discussions about it.
We are interested in these results in order to see, in Section 2.1, how the quotient of a Lie group can be described. Lee discusses the Homogeneous Space Construction Theorem, which provides a useful fact, but only in the case of Lie groups.

## 1.5 Riemannian manifolds

This section is mostly taken from Do Carmo and Flaherty Francis (1992). We now collect some of the important facts that we will need.

**1.5.1 Definition** (Riemannian manifold)**.** A *Riemannian manifold* is a pair $(M, g)$, where $M$ is a smooth manifold and $g$ assigns to each point $p \in M$ smoothly an inner product

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

on the tangent space $T_pM$, varying smoothly with $p \in M$.

**1.5.2 Definition** (Affine connection)**.** An *affine connection* $\nabla$ on a smooth manifold $M$ is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y$$

satisfying for all $X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$:

- $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$ (linearity in the first argument),
- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ (linearity in the second argument),
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y$ (Leibniz rule).

A vector field along a curve $\gamma : I \longrightarrow M$ is a smooth map $X : I \longrightarrow TM$ such that, for any $t \in I$, $X(t) \in T_{\gamma(t)}M$.

**1.5.3 Proposition.** Let $M$ be a smooth manifold with an affine connection $\nabla$. There exists a unique correspondence which associates to a vector field $V$ along the smooth curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along $c$, called the covariant derivative of $V$ along $c$, such that:

(a) $\dfrac{D}{dt}(V + W) = \dfrac{DV}{dt} + \dfrac{DW}{dt}$.

(b) $\dfrac{D}{dt}(fV) = \dfrac{df}{dt}V + f\dfrac{DV}{dt}$, where $W$ is a vector field along $c$ and $f$ is a smooth function on $I$.

(c) If $V$ is induced by a vector field $Y \in \mathfrak{X}(M)$, i.e., $V(t) = Y(c(t))$, then $\dfrac{DV}{dt} = \nabla_{\dot{c}(t)} Y$.

A vector field along curve $\gamma : I \longrightarrow M$ is called parallel when $\frac{DV}{dt} = 0$, for all $t \in I$.

**1.5.4 Theorem** (Existence and uniqueness of the Levi-Civita connection)**.** Let $(M, g)$ be a Riemannian manifold. There exists a unique affine connection $\nabla$ on $M$ such that

- $\nabla$ is *metric-compatible*, i.e. for all vector fields $X, Y, Z$,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

- $\nabla$ is *torsion-free*, i.e. for all $X, Y$,

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

This connection $\nabla$ is called the *Levi-Civita connection*.

A parametrized curve $\gamma : I \longrightarrow M$ is a geodesic if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at $t_0$, for any $t_0 \in I$.
The important fact is that, after solving the ODE in local coordinates, for each $p \in M$ and $v \in T_pM$, there exists a unique geodesic $\gamma_v(t)$, defined in a neighborhood of $0$, such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. For small $v$, this $\gamma_v$ can be defined at $[0, 1]$. Thus, we can define the the exponential map on a small open set $U$ containing $p$,

$$\begin{array}{rcl} \mathrm{Exp}_p\colon T_pM \supseteq U & \longrightarrow & M \\ v & \longmapsto & \mathrm{Exp}_p(v) = \gamma_v(1). \end{array}$$

A Riemannian manifold is called geodesically complete if the maximal domain of definition of all geodesics is $\mathbb{R}$. One can show that compact manifolds are complete. Moreover,

**1.5.5 Theorem** (Hopf–Rinow Theorem Alexandrino et al. (2015))**.** Let $(M, g)$ be a connected Riemannian manifold and $p \in M$. The following statements are equivalent:

(i) $\exp_p$ is globally defined, that is, $\exp_p : T_pM \to M$;

(ii) $M$ is geodesically complete;

(iii) $(M, \mathrm{dist})$ is a complete metric space;

$$dist(p, q) := \inf \left\{ \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt : \ \gamma(a) = p,\ \gamma(b) = q,\ \gamma \text{ piecewise smooth} \right\}.$$

(iv) Every closed bounded set in $M$ is compact.

If $M$ satisfies any (hence all) of the above items, each two points of $M$ can be joined by a minimal geodesic segment. In particular, for each $x \in M$ the exponential map $\exp_x : T_xM \to M$ is surjective.

Next, we recall the concept of curvature.

**1.5.6 Definition** (Riemann curvature tensor)**.** Let $(M, g)$ be a Riemannian manifold with connection $\nabla$. The *Riemann curvature tensor* is the (1,3)-tensor defined by:

$$\forall\ X, Y, Z \in \mathfrak{X}(M) : \quad R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

**1.5.7 Definition** (Sectional curvature)**.** Let $(M, g)$ be a Riemannian manifold, and let $\sigma \subset T_pM$ be a 2-dimensional subspace spanned by linearly independent vectors $u, v \in T_pM$. The *sectional curvature* of $\sigma$ at $p$ is defined by:

$$\kappa(\sigma) = \frac{g(R(u, v)v, u)}{\|u\|^2\|v\|^2 - g(u, v)^2}.$$

**1.5.8 Definition** (Ricci curvature)**.** The *Ricci curvature* is the (0,2)-tensor defined by taking the trace of the Riemann curvature tensor:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y).$$

**1.5.9 Definition** (Scalar curvature)**.** The *scalar curvature* is the function $S$ on $M$ defined as the trace of the Ricci tensor:

$$\rho = \mathrm{tr}_g(\mathrm{Ric}) = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i), \qquad \text{where } \{e_i\} \text{ is a local orthonormal basis.}$$

**1.5.10 Theorem.** [Bonnet–Myers] Let $(M^n, g)$ be a complete Riemannian manifold of dimension $n$. Suppose there exists a constant $\delta > 0$ such that the Ricci curvature satisfies

$$\mathrm{Ric}(v,v) \geq (n-1)\,\delta\,\|v\|^2 \qquad \text{for all } v \in TM.$$

Then:

(i) $M$ is compact,

(ii) the diameter of $M$ satisfies $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\delta}}$,

(iii) and the fundamental group $\pi_1(M)$ is finite.

# 2. Lie groups and Lie Algebras

In this chapter, we discuss general aspects of Lie groups and Lie algebras, with some examples. One important point that we address is how Lie algebras arise from Lie groups.

## 2.1 Lie groups

**2.1.1 Definition** (Lie group)**.** A Lie group $\mathbf{G}$ consists of

(i) a group $G = (G, m, inv, e_G)$, where $m$ is the binary operation and $inv$ the inverse mapping;

(ii) a smooth structure on $G$;
so that:

(iii) $G$ is a smooth manifold;

(iv) $m$ and $inv$ are smooth maps.

Thus, a Lie group is simply a group that is also smooth manifold where the group operations are smooth.
✎ **Notice.** This definition is about the smooth manifolds (real) but one can think also the complex Lie groups where we require to be complex manifold and the operations to be holomorphic. From now on, unless otherwise specified, by a Lie group we mean a real Lie group. [1]

After $\mathbb{R}^n$, the one most important example is the linear group $\mathbf{GL}(n, \mathbb{R})$ and some of its subgroups.

**2.1.2 Example** (Linear Lie group)**.** Identifying $\mathbf{M}_n(\mathbb{R})$ to $\mathbb{R}^{n^2}$, the determinant map, being polynomial, is continuous. Then $\mathbf{GL}(n, \mathbb{R}) = \mathbf{M}_n(\mathbb{R}) \setminus det^{-1}(\{0\})$ is open subset of $\mathbf{M}_n(\mathbb{R})$. So it is manifold of dimension $n^2$. By Remark 1.1.3, it is manifold. Also, it is group under the multiplication of polynomial in the entries; the inverse mapping is given by rational functions with nonzero determinant in the denominator, hence both operations are smooth. Hence $\mathbf{GL}(n, \mathbb{R})$ is a Lie group.

We will prove later that any closed subgroup of $\mathbf{GL}(n, \mathbb{R})$ is also Lie group.

**2.1.3 Example** (Product)**.** As the product of smooth manifolds is smooth manifold and also group, then we should expect that the product of Lie group is again Lie groups. It is. Indeed, the smoothness of the inverse and the binary operation is the direct consequence of Remark part (i) and (ii).

**2.1.4 Definition** (Lie group morphism)**.** A Lie group morphism is just a map between Lie group that is smooth and group homomorphism.
✎ **Notice.** For this definition, continuity is enough, we will prove it later at Proposition 3.1.7 but for some of the examples, we assume it as fact.

The composition of Lie group morphisms is clearly Lie group morphism. Then we can talk about the isomorphism of Lie group and category of Lie groups $\mathcal{LG}$.

**2.1.5 Example** (Projections)**.** We want to see that the product of Lie groups verifies the universal property with projections. By definition, the projections are Lie group morphisms[2]. It is also clear the unique map that for any pair of Lie group morphisms $g : M \longrightarrow G$ and $h : M \longrightarrow H$, the unique map that commutes with the projections $g \times h$ is also Lie group morphism. Hence it is the product in $\mathcal{LG}$.

[1] That is fair because any complex manifold is smooth manifold.
[2] From Remark 1.1.7 part (iii).

**2.1.6 Proposition.** Any Lie group morphism $G \xrightarrow{F} H$ has constant rank. Consequently, any bijective Lie group morphism is an isomorphism.

*Proof.* It is straightforward to see that the translations (left and right) are diffeomorphisms and

$$F(L_p(x)) = L_{F(p)}(F(x)) \implies (dF)_p \circ (dL_p)_{e_G} = (dL_{F(p)})_{e_H} \circ (dF)_e.$$

Then $(dF)_p = (dL_{F(p)})_{e_H} \circ (dF)_e \circ (dL_p)^{-1}_{e_G}$ has same rank than $(dF)_e$ because composition with an isomorphism don't change rank. The last is a consequence of Propositions 1.3.7 and 1.3.8. $\square$

**2.1.7 Definition** (Lie Subgroup)**.** A subgroup $H$ of Lie group $G$ is said immersed Lie subgroup if it is an immersed submanifold of $G$ and the binary operation is smooth map fro $H \times H$ to $H$. Moreover, if the topology that makes $H$ submanifold of $G$ is the one induced from $G$ (embedded submanifold), we say that it is embedded Lie subgroup.
✎ **Notice.** After building the theory, one can prove is that if $H$ is subgroup of $G$ and $H$ is an immersed submanifold of $G$, then, it has to be immersed Lie subgroup. The proof is by following the sketch in Spivak (1970), Problem 20 of chapter 10.

For us, Lie subgroup is always immersed one, unless mention the opposite. We will explain later why.

**2.1.8 Example.** Let $\mathbb{R} \xrightarrow{F} S^1 \times S^1, F(x) = (e^{2\pi i x}, e^{\pi\sqrt{2}ix})$. It is injective and smooth immersion as $F'(p) \neq 0$ for any $p \in \mathbb{R}$. Then $F(\mathbb{R})$ is an immersed submanifold of $S^1 \times S^1$. We know that $S^1$ is multiplicative group(Lie group) and clearly $F$ is (Lie) group homomorphism. Then $F(\mathbb{R})$ is a Lie immersed subgroup of $S^1 \times S^1$.
Moreover, noticing that for any $\varepsilon > 0$, there exists a positive integer $n$ such that the decimal part of $\frac{n\sqrt{2}}{2}$ is less than $\varepsilon$, and that $|e^{ix} - 1| \leq |x|$, we get $|F(n) - F(0)| \leq 2\pi \left(\frac{n\sqrt{2}}{2} - \left\lfloor \frac{n\sqrt{2}}{2} \right\rfloor\right) < 2\pi\varepsilon$. Then there is a sequence $(y_n)_{n\geq 0} \subset F(\mathbb{N}_{\geq 1})$ that converges to $F(0)$. Since there is no sequence in $\mathbb{N}_{\geq 1}$ that converges to $0$, the map $F(\mathbb{R}) \xrightarrow{F^{-1}} \mathbb{R}$ is not continuous, i.e., $F$ is not an embedding. Thus $F(\mathbb{R})$ is not an embedded submanifold.

**2.1.9 Proposition.** Let $H$ be open subgroup of a Lie group $G$. Then it is embedded Lie subgroup of $G$ and union of connected components of $G$.

*Proof.* By Remark 1.1.3, $H$ has a chart induced from $G$. Then, in local coordinates, the inclusion $\hat{i}$ is just the inclusion map, so its derivative is injective at every $h \in H$. Hence, $H$ is an embedded Lie subgroup. Noticing that $G = \bigcup_{g\in G} L_g(H)$, $\quad H = G \setminus \bigcup_{g \in G\setminus H} L_g(H)$, and that each $L_g$ is a homeomorphism, it follows that $H$ is also closed. Since $H$ is both closed and open, it is a union of connected components. Indeed, if $H$ intersects a connected component $C$, then $H \cap C$ is a nonempty, closed, and open subset of the connected space $C$. Therefore, $H \cap C = C$, and hence $C \subset H$. $\square$

It is not hard to see that an open neighbourhood $V$ of $e_G$ generates an open subgroup of $G$. Indeed, $\tilde{V} = V \cup (-V)$ is open, and $\tilde{V} +$[3] $\tilde{V} = \bigcup_{v\in\tilde{V}} L_v(\tilde{V})$ is open. Similarly, $V_k = \underbrace{\tilde{V} + \cdots + \tilde{V}}_{k \text{ times}}$ is open for every $k$. Then $\langle V \rangle$ is open. Using a similar argument and the fact that $e_G \in V, -V, \tilde{V}, L_v(\tilde{V})$, and $V_k$, we also obtain that if $V$ is connected, then $\langle V \rangle$ is connected.

**2.1.10 Proposition.** Let $G$ connected Lie group and $U$ is an open and $e_G \in H$. Then $U$ generates $G$.

*Proof.* From above discussion with Proposition 2.1.9. $\square$

[3] The symbol $+$ denotes the group operation.

Also, one can show that the identity component is an open normal subgroup of $G$, and that it is diffeomorphic to every connected component.

**2.1.11 Proposition.** Let $H$ be an immersed Lie subgroup of $G$. Then $H$ is closed if and only if $H$ is embedded.

*Proof.* Assume that $H$ is embedded. We show that $H$ is closed in $G$. First, we claim that $H$ is open in $\overline{H}$. Indeed, let $p \in H$. Since $H$ is an embedded submanifold, there exists a chart $(U, \phi)$ of $G$ with $p \in U$ such that $\phi(U \cap H) = \tilde{U} \cap \mathbb{R}^k$. Taking closures in $U$ and in $\tilde{U}$, we obtain $\phi(U \cap \overline{H}) = \tilde{U} \cap \mathbb{R}^k = \phi(U \cap H)$, hence $U \cap \overline{H} = U \cap H$. Thus, $U \cap \overline{H}$ is an open neighbourhood of $p$ in $\overline{H}$, and therefore $H$ is open in $\overline{H}$. Next, for each $x \in \overline{H}$, the left translation $L_x$ restricts to a homeomorphism of $\overline{H}$, so the sets $xH = L_x(H)$ are open and dense in $\overline{H}$. Now let $a \in \overline{H}$. Since $a \in aH$ and $aH$ is dense in $\overline{H}$ while $H$ is open in $\overline{H}$, we have $H \cap aH \neq \emptyset$. Hence $aH = H$, which implies $a \in H$. Therefore $\overline{H} \subseteq H$, so $H$ is closed.

The converse is found in Lee (2003). $\square$

**2.1.12 Theorem.** Any closed subgroup $H$ of a Lie group $G$ is always Lie subgroup.

The proof can be found in Spivak (1970), theorem 15 of chapter 10. By Proposition 2.1.11, it is embedded.

This theorem allows us to understand what the linear Lie groups are. They are the subgroups of $\mathbf{GL}(n, \mathbb{R})$[4] which are Lie groups with the topology inherited from $\mathbf{GL}(n, \mathbb{R})$ (and also from $\mathcal{M}_n(\mathbb{R})$). That is, they are the embedded Lie subgroups of $\mathbf{GL}(n, \mathbb{R})$. Hence, the linear Lie groups are the closed subgroups of $\mathbf{GL}(n, \mathbb{R})$.

As examples, there are

$$
\begin{aligned}
&\mathbf{SL}(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) : det(A) = 1\}, \\
&\mathbf{O}(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) : A^T A = I_n\}, \\
&\mathbf{SO}(n, \mathbb{R}) = \mathbf{O}(n, \mathbb{R}) \cap \mathbf{SL}(n, \mathbb{R}), \\
&\mathbf{O}(p, q) = \{A \in \mathbf{GL}(p+q, \mathbb{R})\mathbf{SL}(p+q, \mathbb{R}) : A^T I_{p,q} A = I_{p,q}\}, \\
&\mathbf{SO}(p, q) = \mathbf{O}(p, q) \cap \mathbf{SL}(p+q, \mathbb{R}), \\
&\mathbf{Sp}_{2n}\mathbb{R} = \{A \in \mathbf{GL}(2n, \mathbb{R}) : A^T J_n A = J_n\},
\end{aligned}
$$

where $I_{p,q} = diag(1, \cdots, 1, -1, \cdots, -1)$ ($p$ positive and $q$ negative) and $J_n = (a_{ij})_{1 \leq i,j \leq 2n}$ with $a_{n+i,i} = 1, a_{i,n+1} = -1$ for $i = 1, \cdots, n$ and zero otherwise. We may add some examples later. There are also some complex Lie groups defined in a similar way. Just a remark: the unitary groups $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are defined like $\mathbf{O}(n, \mathbb{R})$ and $\mathbf{SO}(n, \mathbb{R})$ but over $\mathbb{C}$, replacing the transpose by the conjugate transpose. However, they are not complex Lie groups, but real Lie groups.

**2.1.13 Lemma.** Consider the smooth map,

$$
\begin{array}{rcl}
m_k\colon \prod_{i=1}^{i=k} G & \longrightarrow & G \\
(g_1, \cdots, g_k) & \longmapsto & m_k((g_1, \cdots, g_k)) = m(g_1, m(g_2, \cdots, m(g_{k-1}, g_k) \cdots) = g_1 \cdot g_2 \cdots g_k.
\end{array}
$$

Let us agree that whenever we write $\alpha'(0) \in T_pG$, we mean that $\alpha$ is a curve in $G$ such that $\alpha(0) = p$.

[4]We can assume all subgroups of $\mathbf{GL}(n, \mathbb{R})$ because $\mathbf{GL}(n, \mathbb{C})$ can be viewed inside $\mathbf{GL}(N, \mathbb{R})$.

In this case,

$$(dm_k)_{(p_1,\cdots,p_k)}(\alpha_1'(0),\cdots,\alpha_k'(0)) = \sum_{j=1}^{j=k} \frac{d}{dt}\Big|_{t=0} \Big( \prod_{1\le i<j} p_i \cdot \alpha_j(t) \cdot \prod_{j<i\le k} p_i \Big) \tag{2.1.1}$$

$$(\mathrm{d}inv)_e(\alpha'(0)) = -\,\alpha'(0) \tag{2.1.2}$$

$$(\mathrm{d}inv)_g(\alpha'(0)) = -\,(\mathrm{d}R_{g^{-1}})_e \circ (\mathrm{d}L_{g^{-1}})_g. \tag{2.1.3}$$

*Proof.* Recall that $T_{(p_1,\cdots,p_k)}\prod_{i=1}^k G$ can be identified with $\prod_{i=1}^k T_{p_i}G$. To prove (2.1.1), it suffices to find the image of $(0,\cdots,\alpha_j'(0),\cdots,0)$. To find it, consider the curve $(\alpha_i(t))_{i=1}^k$ such that for $i \neq j$, $\alpha_i(t) = p_i$, so that $\alpha_i'(0) = 0 \in T_{p_i}G$. Next, $m_k(\alpha_1(t),\cdots,\alpha_k(t)) = \prod_{1\le i<j} p_i \cdot \alpha_j(t) \cdot \prod_{j<i\le k} p_i$. Hence we get (2.1.1). In particular, $(\mathrm{d}m)_{(e,e)}((u,v)) = u+v$. To get (2.1.2), just notice that $m \circ (inv, \mathrm{Id})$ is a constant map, $(\mathrm{d}(inv,\mathrm{Id}))_e(u) = ((\mathrm{d}inv)_e(u), (\mathrm{d}Id)_e(u))$, and using the chain rule. From these, we obtain the result. For (2.1.3), we apply the chain rule to $inv = R_{g^{-1}} \circ inv \circ L_{g^{-1}}$ at the point $g$. □

Later, we will see that working in a simply connected space is more important than working in a merely connected one. Then we may need a way to pull back a group structure via a covering map.

**2.1.14 Proposition.** Let $G$ be a Lie group, $H$ be a connected manifold and $\pi : H \longrightarrow G$ a covering map[5]. For $e \in \pi^{-1}(e_G)$, there is a unique Lie group structure on $H$ such that $\pi$ is a Lie group morphism, $e$ is the neutral element and the kernel of $\pi$ is a subgroup of the center of $H$.

*Proof.* The task is to define the binary operation $b$ in $H$ and show that it works. $H$ is path connected.

$$\begin{aligned} b\colon H\times H &\longrightarrow H\\ (x,y) &\longmapsto \omega(1), \end{aligned}$$

where $\omega$ is the lifting of $L_{\pi(x)} \circ \pi \circ \beta$, where $\beta$ is a path starting from $e_H$ to $y$.

Now suppose we choose another path $\beta'$ starting from $e_H$ to $y$. We are sure that $\beta' = \beta \star \alpha$, where $\alpha$ is a loop at $y$. Then $\omega'$ is the lifting of $L_{\pi(x)} \circ \pi \circ (\beta \star \alpha) = (L_{\pi(x)} \circ \pi \circ \beta) \star (L_{\pi(x)} \circ \pi \circ \alpha)$, that is $\omega \star$(lifting of $L_{\pi(x)} \circ \pi \circ \alpha$ at $\omega(1)$). [6] So $\omega'(1) = \omega(1)$. Hence, $b$ is well-defined.

Clearly $e_H$ is a neutral element for $b$. We can also see that it is associative. For $x \in H$, we can also check that the endpoint of the lifting of $\mathrm{inv} \circ \pi \circ \alpha$ is its inverse if $\alpha$ is the path joining $e_H$ to $x$.

This construction is nothing else than the lifting of $m \circ (\pi \times \pi)$, which is smooth (because the lift of a smooth map through a local diffeomorphism is smooth) and unique.

Moreover, by construction of $b$, $\pi$ is a group homomorphism. Then $\pi$ is a Lie group morphism.

Let $a \in \ker(\pi)$ and $h \in H$, whose path $\alpha$ is the path starting from $e$ to $h$. Since the kernel is normal, we can define a continuous path $\gamma : I \to \ker(\pi)$, $\gamma(t) = \alpha(t)a(\alpha(t))^{-1}$. This must be constant because the kernel (fiber) of a covering map is discrete. Hence $hah^{-1} = \gamma(1) = \gamma(0) = a$. Thus $ah = ha$. □

**2.1.15 Example** (Universal Cover of $\mathbf{SL}(2,\mathbb{R})$)**.** $\mathbf{SL}(2,\mathbb{R})$ has to be connected because it can be identified with $\mathbf{S} = S^1 \times \mathbb{R}_{>0} \times \mathbb{R}$. Indeed, the Iwasawa decomposition states that every element $M \in \mathbf{SL}(2,\mathbb{R})$

[5] Also smooth.

[6] This second lifting is a loop as $\alpha$ is a loop. To see it, $\pi(y)$ and $L_{\pi(x)}$ are joined by a path, let say $\varepsilon$. Then $(\pi \circ \alpha) \star \varepsilon = \varepsilon \star (L_{\pi(x)} \circ \pi \circ \alpha)$. Checking carefully the right starting point of the lifting, the lifting $\tilde{\varepsilon}$ of $\varepsilon$ is also a loop.

is written uniquely

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad \text{with } \theta \in [-\pi, \pi]/\{-\pi \sim \pi\},\ a > 0,\ u \in \mathbb{R}.$$

Moreover, this identification is a smooth correspondence from **S** to $\mathbf{SL}(2, \mathbb{R})$. Conversely,

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbf{SL}(2, \mathbb{R}) \mapsto \left( \operatorname{atan2}^{7}(m_{21}, m_{11}), \sqrt{m_{11}^2 + m_{21}^2}, \frac{m_{11}m_{12} + m_{21}m_{22}}{m_{11}^2 + m_{21}^2} \right) \in \mathbf{S}.$$

Then $\mathbf{SL}(2, \mathbb{R})$ is diffeomorphic to **S**, which is not simply connected.

Let us call $f : \mathbf{S} \longrightarrow \mathbf{SL}(2, \mathbb{R})$. This identification can give us the universal cover of $\mathbf{SL}(2, \mathbb{R})$, $\tilde{\mathbf{S}} = \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}$. The covering map is

$$\begin{array}{rcl} \pi \colon \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} & \longrightarrow & \mathbf{SL}(2, \mathbb{R}) \\ (\theta, a, u) & \longmapsto & \pi((\theta, a, u)) = \begin{pmatrix} a\cos\theta & au\cos\theta - a^{-1}\sin\theta \\ a\sin\theta & au\sin\theta + a^{-1}\cos\theta \end{pmatrix}. \end{array}$$

Proposition 2.1.14 says $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}$ has a unique group structure to make it a Lie group and $\pi$ a Lie group morphism, if we fix $e = (0, 1, 0) \in \pi^{-1}(I_2)$ the identity. One can define and lift the operation from $\mathbf{SL}(2, \mathbb{R})$, as in the proof. The idea is that there is a group structure $\omega$ on **S** making $f : (\mathbf{S}, \omega) \longrightarrow \mathbf{SL}(2, \mathbb{R})$ an isomorphism of groups. Then, the unique group structure on $\tilde{\mathbf{S}}$ doing the job is the lift of $\omega$.

About another concrete example, one can reconstruct the group law in $\mathbb{R}^n$, if we consider the covering map $\pi : \mathbb{R}^n \longrightarrow T^n$ (torus). But choosing $(0, 0, 0) \in \pi^{-1}(e)$, we recover the classical group operation in $\mathbb{R}^n$.

**2.1.16 Example** (Universal cover of $\mathbf{SO}(3, \mathbb{R})$)**.** Let's start by defining the Pauli matrices

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

As they are linearly independent,

$$\begin{array}{rcl} \mathbb{A} \colon \mathbb{R}^3 & \longrightarrow & V = \operatorname{span}_{\mathbb{C}}\{P_1, P_2, P_3\} \\ (x_1, x_2, x_3) & \longmapsto & \mathbb{A}((x_1, x_2, x_3)) = x_1P_1 + x_2P_2 + x_3P_3, \end{array}$$

is an isomorphism. Noticing $P_k^2 = I_2$ and $P_kP_l = -P_lP_k$ for $k \neq l$,

$$\operatorname{Tr}(P_kP_l) = 2\delta_{kl} \implies \operatorname{Tr}(A(x)A(y)) = 2 < x, y > .$$

So, $g : V^2 \longrightarrow \mathbb{R}, (u, v) \mapsto \frac{1}{2}\operatorname{Tr}(uv)$ is an inner product (metric) that makes $\mathbb{A}$ an isometry.

Next, for any $U \in \mathbf{SU}(2) = \{M \in \mathcal{M}_2(\mathbb{C}) : \det(M) = 1, U^\dagger U = I_2\}$, let's define $\overline{U} : V \longrightarrow V, v \mapsto UvU^{-1}$. It is well-defined, because $V$ is the set of all matrices $X$ in $\mathcal{M}_2(\mathbb{C})$ that verifies $X\dagger = X$ and $\operatorname{Tr}(X) = 0$. It is an isomorphism and $g(\overline{U}(v), \overline{U}(w)) = g(v, w)$. Then it is an isometry and $\mathbb{A}^{-1}\overline{U}\mathbb{A}$

[7] $\operatorname{atan2}(y, x)$ is the angle between the $x$-axis and $\vec{OP}$, for $P = \begin{pmatrix} x \\ y \end{pmatrix}$. It is continuous as we identify $-\pi \sim \pi$ and is defined as long as $\begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix}$ is nonzero.

is a linear isometry of $\mathbb{R}^3$. Then we have a map $\tilde{\phi}$ from $\mathbf{SU}(2)$ to $\mathbf{O}(3,\mathbb{R})$. One can check also, it is continuous and $\overline{I_2} = I_3$, then the range of $\tilde{\phi}$ is a subset of $\mathbf{SO}(3,\mathbb{R})$. Hence, we have

$$\begin{array}{rcl} \phi\colon \mathbf{SU}(2) & \longrightarrow & \mathbf{SO}(3,\mathbb{R}) \\ U & \longmapsto & \phi(U) = \tilde{U}. \end{array}$$

Recall that $\mathbf{SU}(2)$ is not a complex Lie group but it is a real Lie group. Its real structure is the universal cover of $\mathbf{SO}(3,\mathbb{R})$. We just need to see $\mathbf{SU}(2)$ is simply connected and $\phi$ is a covering map, the group operation in $\mathbf{SU}(2)$ coincides with the one we construct in Proposition 2.1.14. Being a group morphism is straightforward as

$$\overline{U_1U_2}(v) = \overline{U_1}(\overline{U_2}(v)) \implies \phi(U_1U_2) = \mathbb{A}^{-1}\overline{U_1U_2}\mathbb{A} = (\mathbb{A}^{-1}\overline{U_1}\mathbb{A})(\mathbb{A}^{-1}\overline{U_2}\mathbb{A}) = \phi(U_1)\phi(U_2).$$

Also, $U^\dagger U = I_2$ and $\det(U) = 1$ means $U = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$, $a^2+b^2+c^2+d^2 = 1$ for some $a,b,c,d \in \mathbb{R}$. Then $\mathbf{SU}(2)$ is homeomorphic to $S^3$, which is simply connected. We can prove that it is a covering map but I think it is a heavy computation by the known formula, using quaternion group: Euler–Rodrigues formula. We will not present it this way but use an important theorem to see it.

**2.1.17 Theorem.** Let $\pi : G \longrightarrow H$ be a morphism of connected Lie groups. $\pi$ is a covering map if and only if $(\mathrm{d}\pi)_a$ is an isomorphism, for all $a \in G$.

Proposition 2.1.6 tells us that $(\mathrm{d}\pi)_a$ is an isomorphism, for all $a \in G$ is equivalent $(\mathrm{d}\pi)_a$ being an isomorphism, for for some $a \in G$.

*Proof.* If $\pi$ is a covering map, then it is a local diffeomorphism. So, $(\mathrm{d}\pi)_a$ is an isomorphism. Conversely, assume $(\mathrm{d}\pi)_a$ is an isomorphism for any $a \in G$. By the Inverse Function Theorem, there are $V$ and $W$ neighbourhoods of the identity of $G$ and of $H$ such that $\pi$ restricted on $V$ is a diffeomorphism onto $W$. By connectedness, each element $y \in H$ can be expressed using the elements in $W$, then $y$ is in the range of $\pi$. So $\pi$ is surjective. For any $y \in H$, $\pi^{-1}(y)$ is discrete because otherwise there would be $x \in \pi^{-1}(y)$ such that $\pi$ is constant $(y)$ near $x$, which contradicts the fact that it is locally a diffeomorphism. Let's call $J = \operatorname{Ker}(\pi)$, a discrete set. For any $y \in H$ and $x \in \pi^{-1}(y)$,

$$a \in \pi^{-1}(y) \iff \pi(a) = \pi(x) \iff x^{-1}a \in \operatorname{Ker}(\pi), \text{ hence } \pi^{-1}(y) = xJ.$$

Also, there are $V_x$ and $W_y$ open neighbourhoods of $x$ and $y$ such that $\pi : V_x \longrightarrow W_y$ is a diffeomorphism. Clearly, for any $a \in J$, $aV_x$ is a subset of $\pi^{-1}(W_y)$. If $g \in \pi^{-1}(W_y)$, then $\pi(g) \in W_y$, so $\pi(v) = \pi(g)$ for some $v \in V_x$. So $a := gv^{-1} \in \operatorname{Ker}\pi$ and $g \in aV_x$. Hence, $\pi^{-1}(y) = \bigsqcup_{a\in J} aV_x$. The disjoint union comes from the fact that if there is $g \in aV_x \cap bV_x$, namely $g = au = bv$ for some $u, v \in V_x$, then $\pi(u) = \pi(a)\pi(u) = \pi(b)\pi(v) = \pi(v)$, hence $a = b$. Finally, for any $a \in J$, $\pi : aV_x \longrightarrow W_y$ is also a diffeomorphism as a composition of $L_{a^{-1}}$ and $\pi : V_x \longrightarrow W_y$. $\square$

**2.1.18 Example** (Continue)**.** This theorem allows us to complete Example 2.1.16. We have proven that $\phi$ is a Lie group morphism. We claim first that $T_e\mathbf{SO}(3,\mathbb{R}) = \{M \in \mathcal{M}_3(\mathbb{R}) : M^T = -M\}$. Indeed, if $M^T = -M$, then $\left(e^{tM}\right)^T\left(e^{tM}\right) = e^{tM^T+tM} = e^0 = I_3$. As $\det(e^{tM}) > 0$, then $\gamma(t) = e^{tM}$ is a curve on $\mathbf{SO}(3,\mathbb{R})$, since $\det(e^{tM}) = e^{\operatorname{Tr}(tM)} > 0$, starting at $I_3$. Then $M = \gamma'(0) \in T_eSO(3)$. Conversely, if $M \in T_e\mathbf{SO}(3,\mathbb{R})$, there is a curve $\gamma(t) \in \mathbf{SO}(3,\mathbb{R})$ starting at $I_3$ such that $\gamma'(0) = M$. One can differentiate $\gamma(t)^T\gamma(t) = I_3$, using $\frac{d}{dt}\Big|_{t=0}\gamma(t)^T = \left(\frac{d}{dt}\Big|_{t=0}\gamma(t)\right)^T$ with the product rule in Lemma 2.1.13, in a clever way, we get

$$M + M^T = \gamma'(0)\gamma(0)^T + \gamma(0)^T\gamma'(0)^T = 0 \implies M^T = -M.$$

We do the same reasoning to get $T_e\mathbf{SU}(2) = \{X \in \mathcal{M}_2(\mathbb{C}) : X^\dagger = -X, \det(X) = 1\}$. We will generalize this later, but for now, we compute them directly.

On the other hand, we can write directly

$$\phi(U)(x) = \mathbb{A}^{-1}(U\mathbb{A}(v)U^{-1}) \implies \Big((\mathrm{d}\phi)_e(M)\Big)(x) = \frac{d}{dt}\Big|_{t=0} \mathbb{A}^{-1}(e^{tM}\mathbb{A}(x)e^{-tM}) = \mathbb{A}^{-1}\Big(\frac{d}{dt}\Big|_{t=0} e^{tM}\mathbb{A}(x)e^{-tM}\Big).$$

Then $\Big((\mathrm{d}\phi)_e(M)\Big)(x) = \mathbb{A}^{-1}(M\mathbb{A}(x) - \mathbb{A}(x)M)$.

It is clear that $\{iP_k : k = 1, 2, 3\}$ forms a basis of $T_e\mathbf{SU}(2)$. One can compute $P_kP_l - P_lP_k = 2i\varepsilon_{klm}$[8] $P_m$ for $m \neq k, l$. Then, by linearity,

$$\Big((\mathrm{d}\phi)_e(iP_k)\Big)(x) = \sum_{l,m\neq k,l} 2x_l\mathbb{A}^{-1}(i^2\varepsilon_{kll}P_m) = 2\sum_{l,m\neq k,l} \varepsilon_{lkm}x_le_m = 2x \wedge e_k. \tag{2.1.4}$$

$$\implies (\mathrm{d}\phi)_e(iP_k) = N_k, \quad N_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}, \; N_2 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \; N_3 = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.1.5}$$

Clearly, $(N_k)$ is a basis of $T_e\mathbf{SO}(3, \mathbb{R})$. Then $(\mathrm{d}\phi)_e$ is an isomorphism and $\phi$ is a covering map.

Proposition 2.1.14 provides us with further examples of Lie groups. We have another way, somehow its inverse: assume we have $\Gamma$ a subgroup of a Lie group $H$, when the quotient $H/\Gamma$ is nice enough to be a Lie group.

**2.1.19 Proposition.** Let $H$ be a Lie group with center $Z(H)$ and let $\Gamma$ be a discrete subgroup of $Z(H)$. Then $H/\Gamma$ has a unique group structure making it a Lie group and the quotient map a covering map.

The idea of the proof is to consider the action of $\Gamma$ on $H$ given by $\gamma \cdot h = h\gamma$. This action is free and proper, so the quotient manifold Theorem 1.4.1 is applicable.
Note that if $\Gamma$ is just a normal and closed subgroup, we can still obtain a Lie group structure on the quotient $H/\Gamma$, making the quotient map a Lie group morphism. This follows from a result that we skip, concerning the Homogeneous Space Characterization Theorem, found in Lee (2003) (Theorem 21.18).

## 2.2 Representation of Lie groups

For a group $G$ and a vector space $V$, a representation of $G$ is a group homomorphism $\rho : G \longrightarrow \mathbf{GL}(V)$. Now, in our case the group has more structure, so we also require smoothness in order for it to be a representation of a Lie group. A subspace $W \subseteq V$ is called *$G$-invariant* or a subrepresentation if $\rho(g)W \subseteq W$ for all $g \in G$. The representation is irreducible if its only subrepresentations are $\{0\}$ and $V$. A morphism between representations $(V, \rho)$ and $(W, \rho')$ is a linear map $T : V \to W$ such that $T\rho(g) = \rho'(g)T$ for all $g$. The classical background and constructions (sum, product, tensor, etc.) can be found in some of our references Hall (2013), Fulton and Harris (2013). We just recall Schur's lemma: if $V, W$ are irreducible representations, then any morphism $T : V \to W$ is either zero or an isomorphism.

### 2.2.1 Adjoint representation of Lie group

Proposition 2.1.10 tells that any neighbourhood of the identity of a connected Lie group can determine entirely the Lie group. This leads us to study the tangent space at the identity.

[8] This is the Levi-Civita symbol, which is $0$ if two indices are equal and is the signature of the permutation (klm) otherwise.

Firstly, let $\psi_g$ be the inner automorphism of $G$, $\psi_g(h) = ghg^{-1}$. Next, $\mathrm{Ad}$ is defined as

$$\begin{aligned} \mathrm{Ad}\colon G &\longrightarrow \mathbf{GL}(T_eG) \\ g &\longmapsto \mathrm{Ad}(g) = (d\psi_g)_e. \end{aligned}$$

The fact that $\psi_{gh} = \psi_g \circ \psi_h$ and the functoriality of differentiation ensures that this is a representation of $G$ on its tangent space at the identity. This is called the adjoint representation of $G$.

**2.2.2 Lemma.** Let $\rho : G \longrightarrow H$ be a Lie group morphism. Then

(a) $\rho$ respects the conjugation action, i.e. $\rho \circ \psi_g = \psi_{\rho(g)} \circ \rho$;

(b) $\rho$ respects the adjoint action, i.e. $(d\rho)_e \circ \mathrm{Ad}(g) = \mathrm{Ad}(\rho(g)) \circ (d\rho)_e$.

*Proof.* (a) is straightforward. We use it together with the functoriality of differentiation to get (b). □

We want to see that a Lie group morphism whose domain is connected is uniquely determined by its differential at the identity. To get it, let's define again

$$\mathrm{ad} = (d\,\mathrm{Ad})_e : T_eG \to \mathrm{End}(T_eG) \text{ and } L(X,Y) := \mathrm{ad}(X)(Y). \tag{2.2.1}$$

**2.2.3 Proposition.** The $L : T_eG \times T_eG \longrightarrow T_eG$, defined in (2.2.1), is a skew-symmetric bilinear map and verifies the Jacobi identity[9]. Moreover, if $\rho : G \longrightarrow H$ is a Lie group morphism, then it respects the adjoint action, i.e.

$$(\mathrm{d}\rho)_e(L(X,Y)) = L((\mathrm{d}\rho)_e(X), (\mathrm{d}\rho)_e(Y)).$$

*Proof.* Let's go step by step. First

$$\begin{aligned} &\mathrm{Ad}(g)(\beta'(0)) = \left.\frac{d}{dt}\right|_{t=0} g\beta(t)g^{-1} \in T_eG \\ \Longrightarrow\ &\mathrm{ad}(\alpha'(0))(\beta'(0)) = \left.\frac{d}{ds}\right|_{s=0}\left.\frac{d}{dt}\right|_{t=0} \alpha(s)\beta(t)\alpha(s)^{-1} \in T_e(T_eG) \cong T_eG \qquad (2.2.2) \\ \Longrightarrow\ &\mathrm{Ad}(h)\big(\mathrm{ad}(\alpha'(0))(\beta'(0))\big) = \left.\frac{d}{ds}\right|_{s=0}\left.\frac{d}{dt}\right|_{t=0} h\alpha(s)\beta(t)\alpha(s)^{-1}h^{-1} \in T_eG \\ \Longrightarrow\ &\mathrm{ad}(\gamma'(0))\big(\mathrm{ad}(\alpha'(0))(\beta'(0))\big) = \left.\frac{d}{dr}\right|_{r=0}\left.\frac{d}{ds}\right|_{s=0}\left.\frac{d}{dt}\right|_{t=0} \gamma(r)\alpha(s)\beta(t)\alpha(s)^{-1}\gamma(r)^{-1} \in T_e(T_eG) \cong T_eG. \end{aligned} \tag{2.2.3}$$

Let's see them as derivation. Consider the curves (by abuse of vocabulary) $(I_\varepsilon = (-\varepsilon,\varepsilon))$,

$$\begin{aligned} \delta : I_\varepsilon \times I_\varepsilon &\longrightarrow G, \\ (s,t) &\mapsto \alpha(s)\beta(t)\alpha(s)^{-1}, \end{aligned} \qquad \begin{aligned} \omega : I_\varepsilon \times I_\varepsilon \times I_\varepsilon &\longrightarrow G \\ (r,s,t) &\mapsto \gamma(r)\alpha(s)\beta(t)\alpha(s)^{-1}\gamma(r)^{-1}. \end{aligned}$$

Let $f \in \mathcal{C}^\infty(G)$, we are interesting in what is $\big(\mathrm{ad}(\alpha'(0))(\beta'(0))\big)f$.

[9] see definition (2.3.1)

$$
\begin{aligned}
\big(\operatorname{ad}(\alpha'(0))(\beta'(0))\big)f =& \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f\circ\delta(s,t) = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f\circ\delta(s,t)\\
=& \frac{d}{dt}\Big|_{t=0} F(t) \quad \text{if } F(t) = \frac{d}{ds}\Big|_{s=0} f\circ\delta(s,t) = (\mathrm{d}(f\circ\delta(-,t)))_0\\
=& \frac{d}{dt}\Big|_{t=0}\Big((\mathrm{d}(f\circ m_3\circ(\alpha, c_{\beta(t)}, inv\circ\alpha))_0\Big), \quad \text{where } c_{\beta(t)} \text{ is constant map}\\
=& \frac{d}{dt}\Big|_{t=0}\Big((\mathrm{d}f)_{\beta(t)}\circ(\mathrm{d}m_3)_{(e,\beta(t),e)}(\alpha'(0),0,-\alpha'(0))\Big) \text{ by (2.1.2)}\\
=& \frac{d}{dt}\Big|_{t=0}\Big((\mathrm{d}f)_{\beta(t)}\big(\alpha'(0)\beta(t)-\beta(t)\alpha'(0)\big)\Big) \text{ by (2.1.1)}\\
=& \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\Big(f(\alpha(s)\beta(t)) - f(\beta(t)\alpha(s))\Big). \qquad (2.2.4)
\end{aligned}
$$

This shows the skew symmetry. To see the Jacobi Identity, we do the similar machinery to get

$$
\Big(\operatorname{ad}(\gamma'(0))\big(\operatorname{ad}(\alpha'(0))(\beta'(0))\big)\Big)f = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}\Big(f(\gamma(r)\alpha(s)\beta(t)) - f(\gamma(r)\beta(t)\alpha(s)) + f(\beta(t)\alpha(s)\gamma(r)) - f(\alpha(s)\beta(t)\gamma(r))\Big).
$$

We can compute now and get

$$
\operatorname{ad}(\gamma'(0))\big(\operatorname{ad}(\alpha'(0))(\beta'(0))\big)f + \operatorname{ad}(\alpha'(0))\big(\operatorname{ad}(\beta'(0))(\gamma'(0))\big)f + \operatorname{ad}(\beta'(0))\big(\operatorname{ad}(\gamma'(0))(\alpha'(0))\big)f = 0.
$$

Now, let $\rho : G \longrightarrow H$ Lie group morphism. Equation (2.2.2) gives us

$$
\begin{aligned}
(\mathrm{d}\rho)_e(\operatorname{ad}(\alpha'(0))(\beta'(0))) =& \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\rho(\alpha(s)\beta(t)\alpha(s)^{-1}) \in T_eH\\
=& \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\Big((\rho(\alpha(s)))(\rho(\beta(t)))(\rho(\alpha(s)))^{-1}\Big)\\
=& ad\Big(\frac{d}{ds}\Big|_{s=0}\rho(\alpha(s))\Big)\Big(\frac{d}{dt}\Big|_{t=0}\rho(\beta(t))\Big)\\
=& (\mathrm{d}\rho)_e(\operatorname{ad}(\alpha'(0)))(\mathrm{d}\rho)_e(\operatorname{ad}(\beta'(0))). \qquad (2.2.5)
\end{aligned}
$$

Hence $\rho$ respects the adjoint action. One can also obtain this directly from Lemma 2.2.2(b). □

**2.2.4 Example.** Let $G$ be a linear Lie group. For fixed $A$ in $G$, $\psi_A(M) = AMA^{-1}$. Identifying $T_eG$ by a linear subspace of $T_e\mathbf{GL}(n,\mathbb{R}) = \mathcal{M}_n(\mathbb{R})$, for any $A \in G$ and $X \in T_eG$ :

$$
\operatorname{Ad}(A)(X) = \frac{d}{dt}\Big|_{t=0} A\exp(tX)A^{-1} = \frac{d}{dt}\Big|_{t=0} AX\exp(tX)A^{-1} = AXA^{-1}.
$$

Now, for $Y \in T_eG$:

$$
\begin{aligned}
\operatorname{ad}(Y)(X) =& \frac{d}{dt}\Big|_{t=0}(\exp(tY)X\exp(-tY)) = \Big((\frac{d}{dt}\exp(tY))X\exp(-tY) + \exp(tY)X(\frac{d}{dt}\exp(-tY))\Big)\Big|_{t=0}\\
=& YX - XY.
\end{aligned}
$$

## 2.3 Lie algebras

**2.3.1 Definition** (Lie Algebra)**.** Let $\mathbb{K}$ be a field. A Lie algebra over $\mathbb{K}$ is a vector space $V$ over $\mathbb{K}$ together with a Lie bracket $[-,-]$, namely a skew-symmetric $\mathbb{K}$-bilinear map which satisfies the Jacobi identity

$$\forall X, Y, Z \in V: \quad [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{2.3.1}$$

Now, a Lie algebra morphism is a linear map between Lie algebras that respects the Lie bracket.

**2.3.2 Example.** Proposition 2.2.3 shows us that $T_eG$ with $\mathrm{ad}$ is a Lie algebra and any Lie group morphism induces a Lie algebra morphism by taking its differential.

**2.3.3 Definition.** A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace that is closed under the Lie bracket. An ideal $\mathfrak{k}$ of $\mathfrak{g}$ is a subalgebra such that for any $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$, $[X,Y] \in \mathfrak{k}$.

**2.3.4 Example.** It is straightforward to see that the set of vector fields on a manifold $M$, $\mathfrak{X}(M)$, is a vector space. For $X, Y \in \mathfrak{X}(M)$, $X \circ Y$ may not satisfy the product rule, but $X \circ Y - Y \circ X$ is a vector field on $M$. It is not hard to check that $[X,Y] := X \circ Y - Y \circ X$ is a Lie bracket on $\mathfrak{X}(M)$. Thus we get a Lie algebra.

Now let's assume that $G = M$ is a Lie group. We mean by a left-invariant vector field a vector field $X \in \mathfrak{X}(G)$ such that for any $g, h \in G$,

$$(\mathrm{d}L_g)_h(X_h) = X_{gh} \quad \Big(\text{or simply } (\mathrm{d}L_g)(X) = X\Big).$$

Again, it is clear that the set of left-invariant vector fields $\mathfrak{L}(G)$ forms a linear subspace of $\mathfrak{X}(G)$. Moreover, for $f \in \mathcal{C}^\infty(G)$ and $X, Y \in \mathfrak{L}(G)$,

$$\begin{aligned}(\mathrm{d}L_g)_h([X,Y]_h)f =&([X,Y](f \circ L_g))(h) = \Big(X(Y(f \circ L_g)) - Y(X(f \circ L_g))\Big)(h)\\ =&\Big(X((\mathrm{d}L_g)_-(Y_-))\Big)_{gh}(f) - \Big(Y((\mathrm{d}L_g)_-(X_-))\Big)_{gh}(f)\\ =&\Big(X(Y(f))\Big)(gh) - \Big(Y(X(f))\Big)(gh) = [X,Y]_{gh}f.\end{aligned}$$

Thus $[X,Y]$ is also left-invariant. Hence $\mathfrak{L}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$.

Now, there are two Lie algebras associated to a Lie group. We want to see that they are the same.

**2.3.5 Lemma.** For a Lie group $G$ and $v = \alpha'(0) \in T_eG$, $X = X^{(v)}$ defined below in (2.3.2), is the unique left-invariant vector field on $G$ such that $X_e = v$.

$$\begin{aligned} X(f)\colon G &\longrightarrow \mathbb{R}\\ g &\longmapsto X(f)(g) = \frac{d}{dt}\Big|_{t=0} f(g\alpha(t)).\end{aligned} \tag{2.3.2}$$

In other words, $X(g) := (\mathrm{d}L_g)_e(v)$.

*Proof.* Clearly $X_e = v$. Let $g, h \in G$ and $f \in \mathcal{C}^\infty(G)$.

$$\Big((\mathrm{d}L_g)_h(X_h)\Big)f = X_h(f \circ L_g) = \Big(X(f \circ L_g)\Big)(h) = \frac{d}{dt}\Big|_{t=0} f(gh\alpha(t)) = X(f)(gh) = X_{gh}f.$$

Then $X$ is left-invariant. Now let $Y \in \mathfrak{L}(G)$ such that $Y_e = v = X_e$. Take arbitrary $g \in G$.

$$Y_g = Y_{ge} = ((\mathrm{d}L_g)_e(Y_e)) = ((\mathrm{d}L_g)_e(X_e)) = X_{ge} = X_g.$$

Hence a left-invariant vector field is uniquely determined by its value at the identity. □

So far, this lemma tells us that, as sets, $T_eG$ and $\mathfrak{L}(G)$ are equivalent. Moreover,

**2.3.6 Proposition.** The correspondence constructed in Lemma 2.3.5

$$\begin{aligned} \mathcal{F}\colon (T_eG, ad) &\longrightarrow (\mathfrak{L}(G), [-,-]) \\ v = \alpha'(0) &\longmapsto \mathcal{F}(v = \alpha'(0)) = X^{(v)}, \end{aligned}$$

gives us an identification (isomorphism of Lie algebras) between $T_eG$ and $\mathfrak{L}(G)$.

*Proof.* The bijection is trivial. We use the uniqueness in Lemma 2.3.5 to get linearity. Now let $v = \alpha'(0)$, $w = \beta'(0)$ and $X = \mathcal{F}(v)$, $Y = \mathcal{F}(w)$. Using (2.3.2), for any $f \in \mathcal{C}^\infty(G)$ we have

$$\begin{aligned} [X,Y]_e f =& \Big(X(Y(f)) - Y(X(f))\Big)(e) = \Big(\frac{d}{dt}\Big|_{t=0} Y(f)(\alpha(t))\Big) - \Big(\frac{d}{ds}\Big|_{s=0} X(f)(\beta(s))\Big) \\ =& \Big(\frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0} f(\alpha(t)\beta(s))\Big) - \Big(\frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f(\beta(s)\alpha(t))\Big) \\ =& \big(\mathrm{ad}(v)(w)\big)f \quad \text{by (2.2.4)}. \end{aligned}$$

Hence $\mathcal{F}\big(\mathrm{ad}(v)(w)\big) = [\mathcal{F}(v), \mathcal{F}(w)]$. □

The composition of Lie algebra morphisms is clearly a Lie algebra morphism. Then we can talk about isomorphisms of Lie algebras and the category of Lie algebras $\mathcal{LA}$. For example, Proposition 2.3.6 gives the isomorphism between $(T_eG, ad)$ and $(\mathfrak{L}(G), [-,-])$. On the other hand, we have established a functor (it clearly respects composition and identity)

$$\begin{aligned} \mathbb{T} : \mathcal{LG} &\to \mathcal{LA} \\ G &\mapsto \mathbb{T}(G) = (T_eG, ad) \\ G \xrightarrow{\rho} H &\mapsto T_e \xrightarrow{(\mathrm{d}\rho)_e} T_eH. \end{aligned} \tag{2.3.3}$$

**2.3.7 Remark** (Product)**.** Given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, one can define a bracket on $\mathfrak{g} \times \mathfrak{h}$ by

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

Clearly, this is a Lie bracket and makes the projections Lie algebra morphisms. Moreover, it is straightforward to see that for any pair of Lie algebra morphisms $g : \mathfrak{a} \longrightarrow \mathfrak{g}$ and $h : \mathfrak{a} \longrightarrow \mathfrak{h}$, the unique map that commutes with the projections $g \times h$ is also a Lie algebra morphism. Hence, it is the abstract product of Lie algebras.

The functor $\mathbb{T}$ respects products because $T_{(e,e)}(G \times H) = T_eG \times T_eH$ and for any $\alpha'(0) = (\alpha_1'(0), \alpha_2'(0))$ and $\beta'(0) = (\beta_1'(0), \beta_2'(0))$ in $T_{(e,e)}(G \times H)$, using (2.2.2),

$$\begin{aligned} [\alpha'(0), \beta'(0)] =& \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} (\alpha_1(s), \alpha_2(s))(\beta_1(t), \beta_2(t))(\alpha_1(s), \alpha_2(s))^{-1} \\ =& \Big(\frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} \alpha_1(s)\beta_1(t)\alpha_1(s)^{-1}, \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} \alpha_2(s)\beta_2(t)\alpha_2(s)^{-1}\Big) \\ =& ([\alpha_1'(0), \beta_1'(0)], [\alpha_2'(0), \beta_2'(0)]). \end{aligned}$$

## 2.4 Correspondence between Lie groups and Lie algebras

So far, we have the functor $\mathbb{T}$, defined at (2.3.3), from the category of Lie groups to the category of Lie algebras. Also, it is straightforward to see that Lie subgroups correspond to Lie subalgebras by considering the inclusion and differentiating it, giving an injective Lie algebra morphism.

**2.4.1 Proposition.** If $H$ is a Lie subgroup of $G$, then $\mathfrak{h} = T_eH$ is a Lie subalgebra of $\mathfrak{g}$.

We also want to see the converse of this fact: given a Lie algebra, is there a Lie group that corresponds to it? Given a Lie algebra $\mathfrak{g}$ coming from a Lie group $G$, what information can we get about $G$ from $\mathfrak{g}$? We will see that there is a correspondence between Lie subalgebras and Lie groups, under a small adjustment.
This objective provides a more convincing definition of a Lie subgroup. We know that $T_e(S^1 \times S^1) = \mathbb{R}^2$ and it is straightforward to see that the bracket from an abelian group is trivial. Then the subalgebras are all the lines passing through the origin. One can show, in the same way we did with Example 2.1.8, that only those with rational slope are embedded. Then, the correct definition of Lie subgroup should be the immersed one. So, why have we fixed this? If we say only Lie subgroup, we mean immersed.

This part is just motivation, but we need to study the exponential map to have tools for the completion of the correspondence.

A representation of a Lie group is a Lie group morphism from $G$ to $\mathbf{GL}(V)$. This induces a Lie algebra morphism from $T_eG$ to $\mathfrak{gl}(V)$, which we call a representation of $T_eG$. More generally,

**2.4.2 Definition.** For a Lie algebra $\mathfrak{g}$ and a vector space $V$, a representation of $\mathfrak{g}$ is a Lie algebra morphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

**2.4.3 Proposition.** Let $(\rho, V)$ be a representation of a Lie group $G$ and $(\varphi, V)$ the induced representation of the Lie algebra $T_eG$. Then $(\rho, V)$ is irreducible if and only if $(\varphi, V)$ is irreducible.

# 3. Exponential map

In this chapter, we construct the exponential map on a Lie group and study its properties. This is a totally algebraic definition, without choosing a metric. We are interested in situations where this exponential map coincides with the exponential map in a general Riemannian manifold. This occurs in certain cases where the Lie group admits a bi-invariant metric. For left-invariant metrics only, the situation remains interesting, since one can define Cartan connections which, as we will see, give rise to many important classes of curvature.

## 3.1 Exponential map and its first consequences

**3.1.1 Definition.** Let $G$ be a Lie group. A one-parameter subgroup of $G$ is a Lie group morphism $\gamma : \mathbb{R} \longrightarrow G$.

First, we recall that for any vector field $X$ on a manifold $M$, $p \in M$ and $v \in T_pM$, there is a unique smooth curve $\alpha : I_\varepsilon = (-\varepsilon, \varepsilon) \longrightarrow M$ such that

$$\alpha(0) = p; \quad \alpha'(0) = v \quad \text{and } X(\alpha(t)) = \alpha'(t).$$

Namely, $\alpha$ is an integral curve of $X$ starting at $p$ with initial velocity $v$.
Also, another fact is that such a path can be extended smoothly to all $t \in \mathbb{R}$ if $X$ is a left-invariant vector field on a Lie group $G$.

**3.1.2 Lemma.** Let $\gamma : I \longrightarrow G$ be a smooth curve such that $0 \in I$ and $\gamma(0) = e$. $\gamma$ is a one-parameter subgroup if and only if $\gamma$ is an integral (maximal) curve of a left-invariant vector field.

*Proof.* Let $\gamma$ be a one-parameter subgroup. We consider $X(g) = (\mathrm{d}L_g)_e \gamma'(0)$. Then $X$ is left-invariant and $X(\gamma(t)) = (\mathrm{d}L_{\gamma(t)})_e \gamma'(0) = \gamma'(t)$. Conversely, assume that $\gamma : I \longrightarrow G$ is an integral curve of a left-invariant vector field $X$. We shall first prove that $\gamma(s+t) = \gamma(s)\gamma(t)$. To show it, fix $s$, and consider $\alpha(t) = \gamma(s+t)$ and $\beta(t) = \gamma(s)\gamma(t)$. Clearly $\alpha(0) = \beta(0) = \gamma(s)$; also $X(\alpha(t)) = \alpha'(t)$ and $\beta'(t) = (\mathrm{d}L_{\gamma(s)})_{\gamma(t)} X(\gamma(t)) = X(\gamma(s)\gamma(t))$. The uniqueness of integral curves starting at $\gamma(s)$ with the same velocity gives us the equality $\alpha = \beta$. Now, if we assume $I = (a, b)$ (or $I = (a, b]$, etc.), take $s = b - \varepsilon \in I$ for very small $\varepsilon$ and we can define $\gamma(s + 2\varepsilon) = \gamma(s)\gamma(2\varepsilon)$. In particular, we can extend $\gamma$ to have $I$ unbounded above. The same reasoning shows that $I$ is unbounded below. By maximality, $I = \mathbb{R}$. $\square$

This lemma implies that for any $v \in T_eG$ there is a unique one-parameter subgroup $\gamma_v$ such that $\gamma_v'(0) = v$. This important fact leads us to the definition of the exponential map.

**3.1.3 Definition.** The exponential map on a Lie group $G$ is defined as

$$\begin{array}{rcl} \exp = \exp_G \colon \mathfrak{g} = T_eG & \longrightarrow & G \\ v & \longmapsto & \exp = \exp_G(v) = \gamma_v(1). \end{array}$$

**3.1.4 Proposition** (Naturality of the exponential map)**.** For any $\rho : G \longrightarrow H$, Lie group morphism,

$$\exp_H \circ (\mathrm{d}\rho)_e = \rho \circ \exp_G .$$

*Proof.* We note that, for $v \in T_eG$, $\rho \circ \gamma_v$ is a one-parameter subgroup of $H$ with initial velocity $(\mathrm{d}\rho)_e(v)$. By uniqueness, $\gamma_{(\mathrm{d}\rho)_e(v)} = \rho \circ \gamma_v$. Thus the result. $\square$

Also, one can get

$$(\mathrm{d}\exp)_0(u) = \left.\frac{d}{dt}\right|_{t=0} \gamma_{tu}(1) = \left.\frac{d}{dt}\right|_{t=0} \gamma_u(t) = \gamma_u'(0) = u.$$

That means $(\mathrm{d}\exp)_0$ is the identity, hence an isomorphism. So, there are open sets $V$ of $\mathfrak{g}$ containing $0$ and $W$ of $G$ where $\exp_G : V \longrightarrow W$ is a diffeomorphism.

**3.1.5 Corollary.** If $G$ is connected, any Lie group morphism $\rho : G \longrightarrow H$ is uniquely determined by its differential at the identity $(\mathrm{d}\rho)_e$.

*Proof.* Since there are open sets $V$ of $\mathfrak{g}$ containing $0$ and $W$ of $G$ where $\exp_G : V \longrightarrow W$ is a diffeomorphism, the naturality of the exponential map gives uniquely the value of $\rho$ on $W$ from $(\mathrm{d}\rho)_e$. Hence Proposition 2.1.10 gives us the conclusion. □

Next, let's try to see some obvious fact about the exponential map. For any $v \in T_eG$ and $\lambda \in \mathbb{R}$, $\left.\frac{d}{dt}\right|_{t=0} \gamma_v(\lambda t) = \lambda v$. By uniqueness, $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. So, $t \mapsto \exp(tv)$ is a one-parameter subgroup of $G$, for any $v \in T_eG$. Conversely, if $\mathbb{E} : T_eG \longrightarrow G$ is a smooth map such that the differential at $0$ is the identity and for any $v \in T_eG$, $\mathbb{E}_v(t) := \mathbb{E}(tv)$ is a one-parameter subgroup, we have $\mathbb{E}_v'(0) = v$ and

$$\exp_G(v) = \mathbb{E}_v(1) = \mathbb{E}(v).$$

**3.1.6 Example.** Recall that for any $M \in \mathcal{M}_n(\mathbb{R})$[1], $||M^k||_2 \leq ||M||_2^k$. Then the norm $e^M = \sum_{k\geq 0} \frac{1}{k!} M^k$ is bounded by $e^{||M||_2}$, which tells us the well-defined definition of exponential of matrices. We also know that it verifies $\det(e^A) = e^{\mathrm{Tr}\,(A)} \neq 0$. Then $e^M \in \mathbf{GL}(n, \mathbb{R})$. Considering it as a map from $T_e\mathbf{GL}(n, \mathbb{R}) = \mathbf{M}_n(\mathbb{R})$ to $\mathbf{GL}(n, \mathbb{R})$, it satisfies that the differential at $0$ is the identity and for any $M \in \mathbf{M}_n(\mathbb{R})$, $t \mapsto e^{tM}$ is a one-parameter subgroup. So, the exponential map for $\mathbf{GL}(n, \mathbb{R})$ is just the exponential of matrices. The same reasoning is valid for all linear Lie groups.

**3.1.7 Proposition.** Let $F : G \longrightarrow H$ be a group morphism between Lie groups. If $F$ is continuous, then $F$ is smooth.

*Proof.* Let $\gamma : \mathbb{R} \longrightarrow H$ be a continuous group morphism. Let $W$ be an open neighbourhood of $0$ in $T_eH$ that makes the restriction of $\exp_H$ a diffeomorphism. There is $V$ an open neighbourhood of $0$ such that $3V \subseteq W$. Also there is $t_0 \in \mathbb{R}^*$ such that $\gamma([-t_0, t_0]) \subseteq \exp_H(V)$. Then $\gamma(t_0) = \exp_H(x_0)$ for some $x_0 \in V$. As $\exp_H$ is a diffeomorphism, $\exp_H^{-1}$ exists on $2\exp_H(V)$,

$$\begin{aligned}
\exp_H\left(2\exp_H^{-1}(\gamma(\tfrac{t_0}{2}))\right) =& \gamma(t_0) = \exp(x_0) \implies \gamma(\tfrac{t_0}{2}) = \exp(\tfrac{x_0}{2}) \in \exp(V) \\
\implies \exp_H\left(2\exp_H^{-1}(\gamma(\tfrac{1}{2}\tfrac{t_0}{2}))\right) =& \gamma(\tfrac{t_0}{2}) = \exp(\tfrac{x_0}{2}) \implies \gamma(\tfrac{t_0}{4}) = \exp(\tfrac{x_0}{4}) \in \exp(V) \\
\implies \gamma(\tfrac{t_0}{2^n}) =& \exp(\tfrac{x_0}{2^n}) \quad \forall n \in \mathbb{N} \\
\implies \gamma(\tfrac{m}{2^n}t_0) =& \left(\gamma(\tfrac{t_0}{2^n})\right)^m = \left(\exp(\tfrac{x_0}{2^n})\right)^m = \exp(\tfrac{m}{2^n}x_0) \quad \forall m \in \mathbb{N} \\
\implies \gamma(st_0) =& \exp(sx_0) \quad \forall s \in \mathbb{R} \quad \text{by density of } \left(\tfrac{m}{2^n}\right)_{m,n} \in \mathbb{R} \\
\implies \gamma(t) =& \exp(\tfrac{x_0}{t_0}t) \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Hence $\gamma$ is smooth. Now let $(x_i)_{i=1}^d$ be a basis of $T_eG$ and define a smooth map,

$$\begin{aligned}
\varepsilon \colon \mathbb{R}^n &\longrightarrow G \\
u &\longmapsto \varepsilon(u) = \exp_G(u_1x_1)\exp_G(u_2x_2)\cdots\exp_G(u_nx_n).
\end{aligned}$$

[1] Similarly for $\mathcal{M}_n(\mathbb{C})$.

One can compute that $(\mathrm{d}\varepsilon)_0(u) = \frac{d}{dt}\Big|_{t=0} \varepsilon(tu) = u_1X_1 + u_2X_2 + \cdots + u_nX_n$. Then $(\mathrm{d}\varepsilon)_0$ is an isomorphism. Then there are neighbourhoods $U$ and $V$ of $0$ and $e$ where $\varepsilon$ is a diffeomorphism. On the other hand $F \circ \varepsilon = F_1F_2\cdots F_n$, where each $F_k(u) = F(\exp(u_kx_k))$ is a smooth map, as it is continuous. Then $F \circ \varepsilon$ is smooth. Near the identity, $F = (F \circ \varepsilon) \circ \varepsilon^{-1}$ is smooth, precisely on $V$. Using the fact that $L_g$ is a diffeomorphism, $F$ is smooth everywhere. □

Moreover, the exponential map is also a tool to understand the correspondence of subgroups. One can see that if $H$ is a Lie subgroup of $G$ then

$$T_eH = \{v \in T_eG : \forall t \in \mathbb{R} : \exp_G(tv) \in H\}. \tag{3.1.1}$$

If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g} = T_eG$, the exponential map can tell us a subgroup of $G$ which corresponds to $\mathfrak{h}$. To see it, let's start with some important facts.

**3.1.8 Theorem.** If $G$ is a Lie group and $X, Y \in T_eG$, then

(a) $\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right)$;

(b) $\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) = \exp\left(t^2[X,Y] + O(t^3)\right)$;

(c) $\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + O(t^3)\right)$.

The proof can be found in Spivak (1970), Theorem 14 of Chapter 10.

**3.1.9 Theorem** (Baker–Campbell–Hausdorff)**.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g} = T_eG$ and exponential map $\exp : \mathfrak{g} \to G$. For $X, Y \in \mathfrak{g}$ sufficiently small there exists a unique element $Z \in \mathfrak{g}$ such that

$$\exp(X)\,\exp(Y) = \exp(Z).$$

Also:

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] + \cdots,$$

and the RHS is a convergent series.

This is a classical result, we will not prove here. It can be seen in any reference that contains Lie theory.

The naturality of the exponential map is very powerful. There are small neighbourhoods $U$ of $e_H$ and $V$ of $e_G$ where $\exp_G : \tilde{V} \longrightarrow V$ and $\exp_H^{-1} : U \longrightarrow \tilde{U}$ are diffeomorphisms. From the naturality, if a Lie group morphism $\varphi : H \longrightarrow G$ is injective, then $(\mathrm{d}\varphi)_e$ is injective on $\tilde{U}$, hence injective on all $T_eH$. By Proposition 2.1.6, it is an immersion. Consequently, $\varphi(H)$ is a Lie subgroup of $G$.

**3.1.10 Theorem.** Let $G$ be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $T_eG$. Then the subgroup $H$, generated by $\exp_G(\mathfrak{h})$, is a Lie subgroup of $G$ and its tangent space is exactly $\mathfrak{h}$.

*Proof.* Let $H$ be the subgroup generated by $\exp_G(\mathfrak{h})$ in $G$. Let $\Delta$ be a small ball centred at the origin in $\mathfrak{g}$ where the $BCH$ formula is valid and $\exp_G : \Delta \longrightarrow \exp(\Delta)$ is a diffeomorphism. Notice that $H$ is generated by a neighbourhood of $e$, $H_0 = \exp(\Delta \cap \mathfrak{h})$. Since $H_0^{-1} = H_0$, we have $H = \bigcup_{k\in\mathbb{N}} H_0^k$. We can put a topology $\tau_0$ on $H_0$ that makes $\Delta \cap \mathfrak{h}$ homeomorphic to $H_0$, via $\exp_G$ (this must be the one induced from $G$). One can also notice that $H_0H_0 \cap \exp(\Delta) = H_0$ (recursively $H \cap \exp(\Delta) = H_0$).

$H$ has a topology such that $\tau_0$ is the induced topology on $H_0$ and all left translations $L_h$ are homeomorphisms. With this topology, there is a unique smooth structure on $H$ in which the translations are also diffeomorphisms and the structure restricted to $H_0$ is the same as the one from $G$. Noticing that $\mathfrak{h}$ is a submanifold of $\mathfrak{g}$, $\Delta \cap \mathfrak{h}$ is a submanifold of $\Delta$, hence $H_0$ is a submanifold of $\exp(\Delta)$, and thus a submanifold of $G$. So the differential of the inclusion $H \hookrightarrow G$ is injective at the points of $H_0$, and therefore at any point, since translations are diffeomorphisms. Moreover, clearly $T_eH = T_eH_0 = \mathfrak{h}$. $\square$

Now, one can ask whether any Lie algebra morphism between tangent spaces of Lie groups comes from a Lie group morphism. In other words, is the functor $\mathbb{T}$ defined at (2.3.3) full?

From Example 2.1.18, $(\mathrm{d}\phi)_e$ is an isomorphism from $T_e\mathbf{SU}(2)$ to $T_e\mathbf{SO}(3,\mathbb{R})$. Then $f = (\mathrm{d}\phi)_e^{-1}$ is an isomorphism from $T_e\mathbf{SO}(3,\mathbb{R})$ to $T_e\mathbf{SU}(2)$. If it comes from a Lie group morphism $\varphi : \mathbf{SO}(3,\mathbb{R}) \longrightarrow \mathbf{SU}(2)$, $\varphi$ has to be a covering map by Theorem 2.1.17. That is a contradiction because $\mathbf{SU}(2)$ is simply connected but $\mathbf{SO}(3,\mathbb{R})$ is connected but not simply connected. Hence, $\mathbb{T}$ is not full.

**3.1.11 Theorem** (Lie's Second Theorem)**.** Let $G, H$ be two Lie groups, with $G$ simply connected, and let $\phi : T_eG \longrightarrow T_eH$ be a Lie algebra morphism. Then there exists a unique Lie group morphism $\varphi : G \longrightarrow H$ such that $(\mathrm{d}\varphi)_e = \phi$.

*Proof.* The graph of $\phi$, $\mathrm{graph}(\phi)$, is clearly a subspace of $T_{(e,e)}(G \times H) = T_eG \times T_eH$. It is closed under the bracket, since $\phi$ is a Lie algebra morphism, hence it is a Lie subalgebra. By Theorem 3.1.10, there exists $K$, a Lie subgroup of $G \times H$, such that $T_{(e,e)}K = \mathrm{graph}(\phi)$.

The differential of the projection $\pi_1 : K \longrightarrow G$ at the identity can be identified with the projection $\tilde{\pi}_1$ of $\mathrm{graph}(\phi)$ onto $T_eG$, which is an isomorphism. Then $\pi_1$ is a covering map, and necessarily an isomorphism of Lie groups because $G$ is simply connected. Now define $\varphi = \pi_2 \circ \pi_1^{-1}$, whose differential can be identified with $\tilde{\pi}_2 \circ \tilde{\pi}_1^{-1} = \phi$. $\square$

The main ideas of the proofs of Theorems 3.1.10 and 3.1.11 are mostly taken from Fulton and Harris (2013). We have tried to provide details and solve some steps that the author left as exercises. As a comment, from Theorem 3.1.10, the connected component of the identity of such $H$ is the unique connected Lie subgroup $H'$ that satisfies $T_eH' = \mathfrak{h}$ because $H'$ has to contain $\exp(\mathfrak{h})$. Another important consequence is Lie's third theorem. First, let's state Ado's theorem.

**3.1.12 Theorem** (Ado's Theorem)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{K}$ of characteristic zero (e.g., $\mathbb{R}$ or $\mathbb{C}$). Then there exists a finite-dimensional vector space $V$ over $\mathbb{K}$ and an injective Lie algebra homomorphism

$$\varphi : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V).$$

The proof of this theorem is given as an appendix in Fulton and Harris (2013), Appendix **E**.

**3.1.13 Theorem** (Lie's third theorem)**.** Every finite-dimensional real Lie algebra $\mathfrak{g}$ is the Lie algebra of some simply connected Lie group.

*Proof.* Ado's theorem tells us that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$, which is a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^n) = T_e\mathbf{GL}(n,\mathbb{R})$. By Theorem 3.1.10, there is a connected Lie subgroup $H$ of $\mathbf{GL}(n,\mathbb{R})$ such that $T_eH = \mathfrak{h}$. Then the universal cover of $H$ is the candidate. $\square$

We get now that the functor $\mathbb{T}$ induces an isomorphism of categories between the category of simply connected Lie groups $\mathcal{SCLG}$ and the category of finite-dimensional Lie algebras $\mathcal{FLA}$.

## 3.2 Some words on classifications

In this part, we discuss some initial classification of Lie algebras. Basically, we just state some definitions with main propositions and some proofs.
If we ask ourselves when a connected Lie group $G$ is simple (has no non-trivial normal subgroup), it is not always easy to check. But the Lie correspondence allows us to study the associated Lie algebra.
First, let $H$ be a connected Lie subgroup of $G$. If it is normal, then for any $v \in T_eH$ and $w \in T_eG$,

$$\gamma(t,s) = \exp_G(sw)\exp_G(tv)\exp_G(-sw) \in H, \gamma(0) = e \implies [u,v] = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\gamma(s,t) \in T_eH.$$

So $T_eH$ is an ideal of $T_eG$. The converse is also true. Indeed, let $T_eH$ be an ideal of $T_eG$ and $u \in T_eH$, $a \in G$. Considering the conjugation $\psi_a : H \to G$ and the naturality of the exponential map we get

$$a\exp_H(tX)a^{-1} = \exp_G\left(Ad(a)x\right) \in H.$$

So $aHa^{-1} \subset H$ holds in a neighbourhood of $H$, and by connectedness it holds on all $H$.
Now, the natural way to define a simple Lie algebra is a Lie algebra with no non-trivial ideals.

Another trivial fact is that if $G$ is abelian then $T_eG$ has bracket zero everywhere. Moreover, if $T_eG$ has bracket zero then $\exp_G(u)\exp_G(v) = \exp_G(u+v) = \exp_G(v+u) = \exp_G(v)\exp_G(u)$. Then $G$ commutes in a neighbourhood of $G$. Again, by connectedness, it commutes everywhere. So the right definition of an abelian Lie algebra is having zero bracket.

There are some classes of Lie algebras still interesting even though they are neither simple nor abelian.

**3.2.1 Definition.** Let $\mathfrak{g}$ be a Lie algebra. The lower central series $\mathcal{D}_k\mathfrak{g}$ and the derived series $\mathcal{D}^k\mathfrak{g}$ are defined recursively as follows: $\mathcal{D}_0\mathfrak{g} = \mathcal{D}^0\mathfrak{g} = \mathfrak{g}$,

$$\mathcal{D}_k\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_{k-1}\mathfrak{g}] \quad \text{and} \quad \mathcal{D}^k\mathfrak{g} = [\mathcal{D}^{k-1}\mathfrak{g}, \mathcal{D}^{k-1}\mathfrak{g}].$$

Clearly $\mathcal{D}^k\mathfrak{g} \subseteq \mathcal{D}_k\mathfrak{g}$. It is also checkable that they are ideals.

**3.2.2 Definition** (Fulton and Harris (2013))**.** (a) We say that $\mathfrak{g}$ is *nilpotent* if $\mathcal{D}^k\mathfrak{g} = 0$ for some $k$.

(b) We say that $\mathfrak{g}$ is *solvable* if $\mathcal{D}^k\mathfrak{g} = 0$ for some $k$.

(c) We say that $\mathfrak{g}$ is *perfect* if $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

(d) We say that $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ has no nonzero solvable ideals.

**3.2.3 Theorem** (Engel's Theorem)**.** Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any Lie subalgebra such that every $X \in \mathfrak{g}$ is a nilpotent endomorphism of $V$. Then there is a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.

*Proof by induction on $n = \dim(\mathfrak{g})$.* For $n = 1$, it suffices to take a basis $X$ and the smallest positive integer $k$ satisfying $X^k = 0$ (this $k$ must be greater than $1$, otherwise $X = 0$). Then there is $v \in V\setminus\{0\}$ such that $X^{k-1}(v) = 0$. This $v$ does the job.
Now assume $\dim(\mathfrak{g}) \geq 2$. Let $\mathfrak{h}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$. $\mathrm{ad}(H)$ induces an endomorphism of $\mathfrak{g}/\mathfrak{h}$ because $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Set $W := \mathfrak{g}/\mathfrak{h}$ and $\mathfrak{k} := \{\overline{\mathrm{ad}}(H) : W \to W, \quad H \in \mathfrak{h}\}$.
Clearly $\mathfrak{k}$ is a linear subspace of $\mathfrak{gl}(W)$ that embeds in $\mathfrak{h}$, hence has dimension at most $\dim(\mathfrak{h}) < \dim(\mathfrak{g})$.
Notice that

$$[\mathrm{ad}(H), \mathrm{ad}(H')](x) = [H,[H',x]] - [H',[H,x]] = \mathrm{ad}([H,H'])(x),$$

for any $x \in \mathfrak{g}$. It induces the same equality in $W$, hence $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{gl}(W)$. By induction, there is nonzero $\overline{X} \in W$ such that for any $K \in \mathfrak{k}$, $K(\overline{X}) = 0$. That means there is $X \in \mathfrak{g} \setminus \mathfrak{h}$ such that for any $H \in \mathfrak{h}$, $\mathrm{ad}(H)(X) \in \mathfrak{h}$. By maximality of $\mathfrak{h}$, $\langle \mathfrak{h}, X \rangle = \mathfrak{g}$, hence $\mathfrak{h}$ is an ideal of codimension $1$.

Now we use induction again to show that the linear subspace

$$U := \{v \in V : \quad \forall H \in \mathfrak{h}, H(v) = 0\} \neq \{0\}.$$

We have $X[U] \subset U$. Indeed, for all $u \in U$ and $H \in \mathfrak{h}$,

$$H(X(u)) = X(H(u)) + [X, H](u) = 0,$$

which implies $X(u) \in U$. As $X$ is nilpotent on $V$, then it is also nilpotent on $U$. So there is $0 \neq v \in U$ such that $X(v) = 0$. $\square$

This gives a characterization of nilpotent Lie algebras $\mathfrak{g}$: it is necessary and sufficient that $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent for any $X \in \mathfrak{g}$.

**3.2.4 Corollary.** A nilpotent Lie algebra $g$ is isomorphic to a subalgebra of the nilpotent Lie algebra in $\mathfrak{gl}(m, \mathbb{R})$ of upper triangular matrices with zeros along the diagonals.

It is a consequence of Ado's theorem and Engel's theorem.

**3.2.5 Theorem** (Lie's theorem, Fulton and Harris (2013))**.** Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a nonzero vector $v \in V$ that is an eigenvector of all $X \in \mathfrak{g}$.

The proof is similar to Engel's theorem. To get an ideal of codimension one, we can consider the quotient map $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, with abelian codomain. Just take a non-trivial linear functional $f : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \longrightarrow \mathbb{K}$ and $\mathfrak{h} = \mathrm{Ker}(f \circ \pi)$ is an ideal of codimension one. By induction the linear subspace $U := \{u \in V : \quad \forall H \in \mathfrak{h}, H(u) = \lambda_H u\} \neq \{0\}$. Now we take $X \in \mathfrak{g} \setminus \mathfrak{h}$ and claim that $X(v) = \lambda u$, for some $u$, to conclude there is $v \neq 0$ in $U$ that is an eigenvector[2] of $X|_W$.

From this theorem, we can say that for any solvable Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, there exists a basis of $V$ where $\mathfrak{g}$ is a Lie subalgebra of upper triangular matrices.

**3.2.6 Proposition.** A Lie algebra $\mathfrak{g}$ is solvable if and only if the Killing[3] form is zero on the commutator $[\mathfrak{g}, \mathfrak{g}]$, if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

**3.2.7 Theorem** (Cartan's Second Criterion)**.** A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.

**3.2.8 Theorem** (Preservation of Jordan Decomposition, Fulton and Harris (2013))**.** Let $\mathfrak{g}$ be a semisimple Lie algebra. For any $X \in \mathfrak{g}$, there exist $X_s$ and $X_n \in \mathfrak{g}$ such that for any representation $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, we have

$$(\varphi(X))_s = \varphi(X_s) \text{ and } (\varphi(X))_n = \varphi(X_n).$$

In other words, if we think of $\rho$ as injective and $\mathfrak{g}$ accordingly as a Lie subalgebra of $\mathfrak{gl}(V)$, the diagonalizable and nilpotent parts of any element $X$ of $\mathfrak{g}$ are again in $\mathfrak{g}$ and are independent of the particular representation $\rho$.

[2] Because we are in the complex case.
[3] See definition 3.3.6.

## 3.3 Relation with the exponential map in Riemannian manifolds

One natural question is why it is called the exponential map. When does it coincide with the definition in a Riemannian manifold, $\mathrm{Exp}_e$? In this section, we will discuss the importance of having a bi-invariant metric, which guarantees that both exponential maps are the same.

Let us first note that if they coincide for some metric $<,>$ on $G$, then $\mathrm{Exp}_e$ is defined everywhere. Hence, by the Hopf–Rinow theorem, it is surjective if $G$ is connected. Therefore, if the exponential map $\exp_G$ is not surjective, we cannot expect to find a Riemannian metric on $G$ such that $\mathrm{Exp}_e = \exp_G$.

The reference that inspired me to write this part is Alexandrino et al. (2015).

**3.3.1 Example.** Recall from Example 3.1.6 that, for a linear Lie group, the exponential map is the matrix exponential. Also, using the relation (3.1.1), any element in $\mathfrak{h} = T_e\mathbf{SL}(2,\mathbb{R})$ is of the form $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. We can see that there is no $A \in \mathfrak{h}$ such that $e^A = X$, where $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Otherwise, $AX = XA$, which implies $a = c = 0$. Then $e^A = A$, a contradiction. This tells us that the exponential map is not surjective. Therefore, we cannot put a metric on $\mathbf{SL}(2,\mathbb{R})$ making the two exponential maps coincide.

One can expect them to be the same if the one-parameter subgroups are the geodesics. In fact, this is true if the metric is bi-invariant.

We say that a metric is left-invariant if the left translations are isometries, i.e.

$$\forall g, h \in G, u, v \in T_hG: \quad \langle (\mathrm{d}L_g)_h(u), (\mathrm{d}L_g)_h(v)\rangle_{gh} = \langle u, v\rangle_h. \tag{3.3.1}$$

Similarly, one defines a right-invariant metric, and a bi-invariant metric is a metric that is both left-invariant and right-invariant. We note that, from any inner product on $T_eG$, we can construct a left-invariant (or right-invariant) metric. Indeed, if $<,>$ is an inner product on $T_eG$,

$$Q_g(u, v) :=< (\mathrm{d}L_{g^{-1}})_g(u), (\mathrm{d}L_{g^{-1}})_g(v) >$$

varies smoothly and satisfies (3.3.1), hence defines a left-invariant metric. However, we cannot always have a bi-invariant metric on an arbitrary Lie group $G$.

**3.3.2 Proposition.** Let $<,>$ be a bi-invariant Riemannian metric on $G$. Then

$$\mathrm{Exp}_e = \exp_G.$$

*Proof.* Let $\nabla$ be the Levi-Civita connection of $<,>$ and let $u, v, w \in T_eG$ with corresponding left-invariant vector fields $X, Y, Z$. The compatibility of $\nabla$ gives

$$\begin{aligned} < \nabla_Y X, Z > &= \frac{1}{2}\Big(X < Y, Z > +Y < Z, X > -Z < X, Y > - < [X,Z], Y > - < [Y,Z], X > - < [X,Y], Z > \Big) \\ &= \frac{1}{2}\Big(- < [X,Z], Y > - < [Y,Z], X > - < [X,Y], Z > \Big). \end{aligned} \tag{3.3.2}$$

Also, $\psi_g = L_g R_{g^{-1}}$, then $\mathrm{Ad}(g) = (\mathrm{d}L_g)_{g^{-1}}(\mathrm{d}R_{g^{-1}})_e$. So

$$\begin{aligned}
& < Ad(\exp(tu))v, Ad(\exp(tu))w >=< v, w > && \text{as the metric is bi-invariant,}\\
\Longrightarrow\ & < [u,v], w > + < v, [u,w] >= 0 && \text{after differentiating.}\\
\Longrightarrow\ & \nabla_Y X = \frac{1}{2}[Y,X] \text{ and } \nabla_X X = 0 && \text{by (3.3.2).}
\end{aligned} \tag{3.3.3}$$

Now, if $\gamma_u$ is the one-parameter subgroup corresponding to $u \in T_eG$ and $X$ is the left-invariant vector field corresponding to $u$, then $X(\gamma_u(t)) = \gamma_u'(t)$. Hence $\frac{D}{dt}\gamma_u'(t) = \nabla_X X = 0$. So $\gamma_u$ is a geodesic. Uniqueness ensures that the geodesics are exactly the one-parameter subgroups (we shall see later that if $\gamma$ is a geodesic, then it is an integral curve of the left-invariant vector field associated with $\gamma'(0)$). □

Thus, the class of Lie groups admitting a bi-invariant metric is an important class of Lie groups. Let us begin by seeing one condition that guarantees the existence of such a metric.

**3.3.3 Proposition.** Every compact Lie group admits a bi-invariant metric $q$.

Before starting the proof, an $n$-form[4] $\omega$ is right-invariant if $(R_h)^*\omega = \omega$ for all $h \in G$, i.e.

$$\forall g, h \in G, u \in (T_gG)^n : \quad \omega_{gh}\big((\mathrm{d}R_h)_g(u_1), \cdots, (\mathrm{d}R_h)_g(u_n)\big) = \omega_g(u_1, \cdots, u_n). \tag{3.3.4}$$

One can show that we can always construct such an $n$-form, since it is clearly uniquely determined by its value at the identity. We can take any alternating multilinear map $\omega_e : (T_eG)^n \longrightarrow \mathbb{R}$ and define

$$\omega_g(u_1, \cdots, u_n) := \omega_e\big((\mathrm{d}R_{g^{-1}})_g(u_1), \cdots, (\mathrm{d}R_{g^{-1}})_g(u_n)\big).$$

It varies smoothly with $g$ and satisfies (3.3.4).

*Proof of Proposition 3.3.3.* Consider a non-vanishing $n$-form $\omega$ and a right-invariant metric $\langle , \rangle$ on $G$. One can define, for any $u, v \in T_aG$,

$$q_a(u,v) := \int_G f_a(g)\,\omega, \quad \text{where } f_a(g) := \langle (\mathrm{d}L_g)_a(u), (\mathrm{d}L_g)_a(v)\rangle_{ga}. \tag{3.3.5}$$

As $G$ is compact, this integral is finite. Also, it is clearly an inner product on $T_aG$ and varies smoothly with $a$. Hence it is a Riemannian metric, and we claim that it is bi-invariant. Remarking that the left and right translations commute, and hence so do their differentials, we quickly get that the metric is right-invariant. To see that it is left-invariant, using $L_{gh} = L_g L_h$,

$$\begin{aligned}
q_{ha}\big((\mathrm{d}L_h)_a(u), (\mathrm{d}L_h)_a(v)\big) &= \int_G \langle L_{gh}u, L_{gh}v\rangle_{gha} = \int_G f_a(gh)\,\omega \\
&= \int_G (f_a \circ R_h)(g)\,(R_h)^*\omega && \text{as } \omega \text{ is right-invariant,}\\
&= \int_G R_h^*(f_a\omega) = \int_G f_a(g)\omega && \text{as } R_h \text{ is a diffeomorphism and } G \text{ is orientable.}
\end{aligned}$$

Hence $q$ is a bi-invariant metric. □

[4] $n$-form $\omega = \big(\omega_x : (T_xG)^n \longrightarrow \mathbb{K}\big)_{x\in G}$, where $\omega_x$ is multilinear, alternating, and $x \mapsto \omega_x$ is smooth (holomorphic for holomorphic $n$-forms).

First, we notice that if the right-invariant metric $\langle , \rangle$ chosen in the proof above is already bi-invariant, then this construction gives the same metric up to a constant.

Now, using (3.3.3), the curvature tensor is computed by

$$R(u,v)w = \frac{1}{4}[[u,v],w] \implies R(u,v,u,v) = \frac{1}{4}||[u,v]||^2. \tag{3.3.6}$$

Hence, the sectional curvature with respect to a bi-invariant metric is always non-negative at the identity, and therefore everywhere by homogeneity (since $L_g$ and $R_g$ are isometries).

**3.3.4 Example** (Sectional curvature on $\mathbf{SO}(3,\mathbb{R})$)**.** First, from (2.1.5), $M_k = -\frac{1}{2}N_k$, the isomorphism

$$\begin{array}{rcl} \mathbb{B}\colon \mathbb{R}^3 & \longrightarrow & T_e\mathbf{SO}(3,\mathbb{R}) = \mathrm{span}_{\mathbb{R}}\{M_1, M_2, M_3\} \\ (x_1,x_2,x_3) & \longmapsto & \mathbb{B}((x_1,x_2,x_3)) = x_1M_1 + x_2M_2 + x_3M_3, \end{array}$$

respects the bracket if we consider the cross product on $\mathbb{R}^3$. Let us take a convenient inner product on $T_eG$. We have the standard one on $\mathbb{R}^3$, which we can push forward to $T_eG$:

$$< x,y >= x\cdot y \implies < u,v >_e =< \mathbb{B}(x), \mathbb{B}(y) >_e := \sum_k x_k y_k = -\frac{1}{2}\operatorname{Tr}(uv).$$

Then the associated right-invariant metric is

$$\langle u,v\rangle_g =< \left(\mathrm{d}R_{g^{-1}}\right)_g(u), \left(\mathrm{d}R_{g^{-1}}\right)_g(v) >_e = -\frac{1}{2}\operatorname{Tr}(ug^{-1}vg^{-1}). \tag{3.3.7}$$

Following what we have learned so far, one can also try to find a right-invariant $3$-form. Again, we choose a convenient alternating trilinear form on $\mathbb{R}^3$, namely the determinant, push it forward to $T_eG$, and obtain

$$\omega_e(u,v,w) = \omega_e(\mathbb{B}(x), \mathbb{B}(y), \mathbb{B}(z)) = \det(x,y,z) = x\cdot(y\wedge z) = -\frac{1}{2}\operatorname{Tr}(u,[v,w]).$$

We should apply (3.3.5) to obtain a bi-invariant metric, but one checks that the metric $\langle , \rangle_g$ defined in (3.3.7) is already bi-invariant. So, with this metric, the exponential maps coincide (at the identity). Moreover, by construction, $M_1, M_2, M_3$ form an orthogonal basis of $T_e\mathbf{SO}(3,\mathbb{R})$, and $[M_k, M_l] = \mathbb{B}(e_k \wedge e_l) = \mathbb{B}(\varepsilon_{klm}e_m) = \varepsilon_{klm}M_m$. Applying (3.3.6),

$$R_{jklm} = R(M_j, M_k, M_l, M_m) = \frac{1}{4}\langle[\varepsilon_{jks}M_s, M_l], M_m\rangle_e.$$

Then

$$R_{jklm} = \frac{1}{4}\sum_{s=1}^{3}\varepsilon_{jks}\varepsilon_{slm} \implies R_{1212} = R_{1313} = R_{2323} = \frac{1}{4}.$$

So the sectional curvature is $\frac{1}{4}$.

But also, a metric that is right-invariant and $\mathrm{Ad}$-invariant must be bi-invariant. In particular, for an abelian group, $\psi_g = Id_G$ and $\mathrm{Ad}(g) = Id_{\mathfrak{g}}$, so we can always find a bi-invariant metric. In other words, $L_g = R_g$, so being right-invariant is equivalent to being left-invariant.
Now, we have two classes of Lie groups that admit bi-invariant metrics. It is straightforward to see that the product metric of bi-invariant metrics is again bi-invariant.
One can ask whether these are all.

**3.3.5 Theorem.** A connected Lie group $G$ admits a bi-invariant metric if and only if it is isomorphic to the Cartesian product of a compact group and a vector space.

The proof of this theorem can be found in Milnor (1976).

In studying the existence of bi-invariant metrics and determining whether the Lie algebra $T_eG$ is semisimple, there is an important tool that we need to introduce: the Killing form.

**3.3.6 Definition** (Killing form)**.** Let $G$ be a Lie group. The Killing form of $\mathfrak{g} = T_eG$ (also called the Killing form of $G$) is the symmetric bilinear form

$$B(u, v) := \mathrm{Tr}\,(\,\mathrm{ad}(u)\,\mathrm{ad}(v)).$$

Since $\mathrm{Ad}(g^{-1})$ is a Lie algebra automorphism, $\mathrm{ad}\,(\,\mathrm{Ad}(g^{-1})(u))\,\mathrm{Ad}(g^{-1})(v) = \mathrm{Ad}(g^{-1})(\,\mathrm{ad}(u)(v))$. Then $\mathrm{Ad}(g) \circ \mathrm{ad}\,(\,\mathrm{Ad}(g^{-1})(u)) \circ \mathrm{Ad}(g^{-1}) = \mathrm{ad}(u)$. So $\mathrm{ad}(\mathrm{Ad}(g)(w)) = \mathrm{Ad}(g) \circ \mathrm{ad}(w) \circ \mathrm{Ad}(g)^{-1}$. This fact implies that the Killing form is $\mathrm{Ad}$-invariant. Consequently, if $G$ is semisimple with negative-definite Killing form $B$, it admits the bi-invariant metric $-B$.

One can also see that, if $G$ admits a bi-invariant metric $Q$, the Ricci tensor does not depend on $Q$. Indeed, $\mathrm{Ric}(u, v) = \mathrm{Tr}\,((R(u, -)v) = \mathrm{Tr}\,(Tr)(\frac{1}{4}[[u, -], v]) = -\frac{1}{4}\,\mathrm{Tr}\,(Tr)([v, [u, -]]) = -\frac{1}{4}B(u, v)$.

**3.3.7 Proposition.** For a semisimple connected Lie group $G$, being compact is equivalent to having a negative-definite Killing form $B$.

*Proof.* See Alexandrino et al. (2015), Theorem 2.35. □

**3.3.8 Proposition** (Alexandrino et al. (2015))**.** Let $G$ be a compact simple Lie group. Then the bi-invariant metric is unique up to rescaling.

## 3.4 Lie Groups with Left-Invariant Metrics and Cartan Connections

In this section, we equip $G$ with a left-invariant metric. Recall that a left-invariant metric is uniquely determined by an inner product at the identity. Then, identifying $T_eG$ with the left-invariant vector fields $\mathcal{L}(G)$, we have

$$< X(g), Y(g) >_g = < X(e), Y(e) >_e .$$

So it is enough to study the inner product on the Lie algebra $\mathfrak{g} = T_eG$.
Recall that equation (3.3.2) is still valid for any left-invariant Riemannian metric. We may need to choose an orthonormal basis $e_1, \cdots, e_n$ and introduce the notation

$$\alpha_{ijk} := < [e_i, e_j], e_k >,$$

so that

$$\nabla_{e_i} e_j = \frac{1}{2}\sum_k (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k.$$

From this, one can compute the sectional curvature

$$\kappa(e_1, e_2) = \sum_k \left(\frac{1}{2}\alpha_{12k}\,(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) - \frac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22}\right).$$

From this equality, we can deduce that if $\mathrm{ad}(e_1)$ is skew-adjoint, then for any $k \neq 1$,

$$\kappa(e_1, e_k) = \sum_j \frac{1}{4}(\alpha_{kj1})^2.$$

This can be generalized: if $\mathrm{ad}(u)$ is skew-symmetric, then for all $v$, $\kappa(u,v) \geq 0$.

**3.4.1 Lemma.** Let $\mathfrak{u}$ be an ideal of dimension $n-1$ and $x \in \mathfrak{g} \setminus \mathfrak{u}$. Let $L$ be the endomorphism of $\mathfrak{u}$ defined by $L(u) := [x, u]$, whose adjoint is denoted by $L^*$[5]. Denoting by $S = \frac{1}{2}(L + L^*)$ the self-adjoint part of $L$, and by $\overline{\nabla}$ the Riemannian connection of the restricted metric (inner product) on $\mathfrak{u}$, for any $u, v \in \mathfrak{u}$,

$$\nabla_x u = \frac{1}{2}(L - L^*)u, \quad \nabla_x x = 0, \quad \nabla_u x = -Su \text{ and } \nabla_u v = \overline{\nabla}_u v + < Su, v > x. \tag{3.4.1}$$

The proof follows mostly from equation (3.3.2).

**3.4.2 Proposition.** Let $x$ be a unit vector of $\mathfrak{g} = T_e G$, orthogonal to the commutator $[\mathfrak{g}, \mathfrak{g}]$. Then the Ricci curvature $r(x)$ is non-positive. It is zero if and only if $\mathrm{ad}(x)$ is skew-adjoint.

*Proof.* Let $\mathfrak{u}$ be the subspace consisting of all $y \in \mathfrak{g}$ such that $< x, y >= 0$. Clearly $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{u}$. Then $\mathfrak{u}$ is an ideal of dimension $n-1$, so we can apply Lemma 3.4.1. Now consider a basis $y_1, \cdots, y_{n-1}$ of $\mathfrak{u}$ consisting of orthonormal eigenvectors of $S$, namely $S(y_i) = \lambda_i y_i$, for each $i = 1, \cdots, n-1$.
As

$$\lambda_i =< S(y_i), y_i >= \frac{1}{2}\big( < L(y_i), y_i > + < L^*(y_i), y_i > \big),$$

we obtain

$$\lambda_i =< L(y_i), y_i >=< y_i, L^*(y_i) > .$$

Using this fact together with (3.4.1), we obtain the sectional curvatures

$$\kappa(x, y_i) = -\lambda_i^2.$$

Hence

$$r(x) = -\operatorname{Tr}(S^2).$$

Moreover, the trace of the square of a self-adjoint operator $S$ is zero if and only if $S = 0$. Therefore, $r(x) = 0$ if and only if $L$ is skew-adjoint. $\square$

A left Haar measure on $G$ is a Radon measure $\mu$ on the Borel $\sigma$-algebra $\mathcal{B}(G)$[6] such that $\mu(gS) = \mu(S)$ for all $g \in G$ and $S \in \mathcal{B}(G)$. It always exists because there is always a nontrivial element in $\bigwedge^n(T_e G)$ (we discussed it in 3.3.3) that induces a left-invariant $n$-form $\omega$, and we can characterize it by

$$\mu(S) = \int_S \omega.$$

It is a Radon measure, left-invariant, and this is the only way to obtain it, namely from an element of $\bigwedge^n(T_e G)$. Since $\bigwedge^n(T_e G)$ is one-dimensional, the left Haar measure is unique up to scaling.

**3.4.3 Definition.** A Lie group $G$ is *unimodular* if the left Haar measure $\mu$ is right-invariant, namely $\mu(Sg) = \mu(S)$ for all $g \in G$ and $S \in \mathcal{B}(G)$. A Lie algebra $\mathfrak{h}$ is unimodular if, for any $h \in \mathfrak{h}$, $\operatorname{Tr}(\mathrm{ad}(h)) = 0$.

It is proven in Helgason (2024) (page $366$) that $G$ is unimodular if and only if the adjoint representation $\mathrm{Ad}(g)$ has determinant $\pm 1$ for every $g \in G$. It follows that semisimple Lie groups are unimodular because $\det(\mathrm{Ad}(g))^2 = 1$, as the nondegenerate Killing form $B$ is $\mathrm{Ad}$-invariant. Another quick consequence of

[5]Recall that it is the unique endomorphism satisfying $< Lu, w >=< u, L^* w >$ for any $u, w \in T_e G$.
[6]$\sigma$-algebra generated by the open sets.

this fact is that compact Lie groups are unimodular. Indeed, the set consisting of the values $|\det(\mathrm{Ad}(g))|$ can be viewed as the image of a continuous group morphism from $G$ to the positive real numbers. Then it is a compact subgroup of $\mathbb{R}_+^*$, so it must be $\{1\}$. The naturality of the exponential map ensures that, for a connected Lie group, $G$ is unimodular if and only if $T_eG$ is unimodular.

**3.4.4 Proposition.** If all Ricci curvatures of a connected Lie group $G$ are non-negative, then $G$ is unimodular.

*Proof.* Let $x \in \mathfrak{g} = T_eG$. Then it is impossible to have $r(x) < 0$. By Proposition 3.4.2, either $x$ is not orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ or $\mathrm{ad}(x)$ is skew-adjoint. But the Jacobi identity gives

$$\mathrm{ad}([g_1, g_2]) = \mathrm{ad}(g_1)\,\mathrm{ad}(g_2) - \mathrm{ad}(g_2)\,\mathrm{ad}(g_1),$$

which implies that every element of $[\mathfrak{g}, \mathfrak{g}]$ belongs to the unimodular kernel

$$\mathfrak{u} = \{u \in \mathfrak{g} : \ \mathrm{Tr}\,(\mathrm{ad}(u)) = 0\}.$$

Then $x$ cannot be orthogonal to $\mathfrak{u}$ unless $\mathrm{Tr}\,(\mathrm{ad}(x)) = 0$. Hence $x \in \mathfrak{u}$. Therefore $\mathfrak{g} = \mathfrak{u}$. $\square$

**3.4.5 Theorem** (Milnor (1976))**.** If a connected Lie group $G$ has a left-invariant metric with all sectional curvatures $\kappa \leq 0$, then it is *solvable*. If $G$ is unimodular, then any such metric with $\kappa \leq 0$ must actually be flat ($\kappa \equiv 0$).

**3.4.6 Example.** Note that the semidirect product of Lie groups has a Lie group structure. In particular, $G = \mathbb{R} \rtimes \mathbb{R}_+^*$ is a Lie group with the law

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 + y_1 x_2,\ y_1 y_2).$$

The neutral element is $e = (0, 1)$ and the inverse of $(x, y)$ is $\left(-\frac{x}{y}, \frac{1}{y}\right)$. Consider the usual inner product $< -, - >$ on $T_eG = \mathbb{R}^2$, and let $Q$ be the corresponding left-invariant metric. We can also check that the differential of the left translation is $\left(\mathrm{d}L_{(a,b)}\right)_{(x,y)} = bI_2$. Then

$$Q_{(a,b)}(u, v) =< \left(\mathrm{d}L_{(-\frac{a}{b}, \frac{1}{b})}\right)_{(a,b)}(u), \left(\mathrm{d}L_{(-\frac{a}{b}, \frac{1}{b})}\right)_{(a,b)}(v) >= \frac{u \cdot v}{b^2}.$$

It is the hyperbolic $\mathbb{H}^2$ metric, so the sectional curvature is $\kappa = -1$. Hence $G$ is solvable.
Another way to see this is to identify it with the set of matrices of the form

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},$$

where $y \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$.
Viewing it as $\mathbb{H}^2$ also allows us to conclude that it is not unimodular.

### 3.4.7 For the three-dimensional case

We assume that $G$ is of dimension 3. Let $< -, - >$ be an inner product on the vector space $T_gG$. For $u, v \in T_gG$, there is a vector $w \in T_gG$ such that

$$< u, w >=< v, w >= 0 \tag{3.4.2}$$

$$|w|^2 = |u|^2 |v|^2 - < u, v >^2 . \tag{3.4.3}$$

First, if $u, v$ are linearly dependent, (3.4.3) forces $w$ to be zero, which also respects (3.4.2). If they are not, the set of all vectors $z$ verifying (3.4.2) is a subspace of $T_gG$, of codimension $2$, hence of dimension one. With the condition (3.4.3), we have two choices $\pm w$. Now, if we fix an orientation on $T_gG$, we have only one choice that makes $(u, v, w)$ oriented. Such a $w$ is the general definition of the cross product $u \times v$.

**3.4.8 Lemma.** Let $G$ be a Lie group. There is a linear endomorphism $L$ of $T_eG$ such that

$$[u, v] = L(u \times v). \tag{3.4.4}$$

Moreover, $L$ is self-adjoint if and only if $G$ is unimodular.

It is because we can choose an oriented orthonormal basis $b_1, b_2, b_3$ and set an endomorphism of $\mathfrak{g}$,

$$L(b_i \times b_j) = [b_i, b_j] = \operatorname{ad}(b_i)(b_j) = -\operatorname{ad}(b_j)(b_i). \tag{3.4.5}$$

It is defined on the basis and extends uniquely to all $\mathfrak{g}$, satisfying (3.4.4). If $L = (l_{ij})_{i,j}$, (3.4.5) also says

$$\operatorname{ad}(b_1)(x_1b_1 + x_2b_2 + x_3b_3) = (l_{31} - l_{21})b_1 + (l_{32} - l_{22})b_2 + (l_{33} - l_{23})b_3,$$

then $\operatorname{Tr}(\operatorname{ad}(b_1)) = -l_{23} + l_{32}$. Doing the same for $b_2, b_3$, we get $\operatorname{Tr}(\operatorname{ad}(b_2)) = l_{13} - l_{31}$ and $\operatorname{Tr}(\operatorname{ad}(b_3)) = -l_{12} + l_{21}$. Hence $L$ is self-adjoint[7] if and only if $\operatorname{Tr}(\operatorname{ad}(x)) = 0$ for any $x \in \mathfrak{g}$, if and only if $\mathfrak{g}$ is unimodular.

Now, let us focus on the case where $G$ is unimodular ($L$ is self-adjoint) and choose an orthonormal basis $e_1, e_2, e_3$ consisting of eigenvectors of $L$, with eigenvalues respectively $\lambda_i$. Then we have

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2 \text{ and } [e_1, e_2] = \lambda_3 e_3. \tag{3.4.6}$$

Following the approach of Milnor, we denote

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i.$$

**3.4.9 Theorem** (Milnor (1976))**.** The orthonormal basis $e_1, e_2, e_3$, chosen as above, diagonalizes the Ricci quadratic form, the principal Ricci curvatures being

$$r(e_1) = 2\mu_2\mu_3, \quad r(e_2) = 2\mu_1\mu_3 \text{ and } r(e_3) = 2\mu_1\mu_2. \tag{3.4.7}$$

Being the trace of the Ricci tensor, the scalar curvature is

$$\rho = 2(\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2). \tag{3.4.8}$$

Milnor (1976) has many nice examples, but I just try another one. Note that $\mathbf{SO}(2, 1)$ is a simple Lie group. Let $G$ be the connected component of the identity, which is then unimodular.

Consider the basis $\{e_1, e_2, e_3\}$ of $T_eG = \mathfrak{so}(2, 1)$ given by:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

[7] We are in the real case.

Their Lie brackets are: $[e_1, e_2] = -e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$. We just need to have an inner product such that $e_1, e_2$, and $e_3$ are orthonormal. Now we consider the left-invariant metric $Q$ corresponding to that inner product, which is $< e_i, e_j >= \delta_{ij}$.
Now, to compute the Ricci curvatures and scalar curvature, we just need to compute $\mu_i$.

$$\mu_1 = -\frac{1}{2}, \qquad \mu_2 = -\frac{1}{2}, \qquad \mu_3 = \frac{3}{2},$$
$$r(e_1) = 2\mu_2\mu_3 = -\frac{3}{2}, \qquad r(e_2) = 2\mu_1\mu_3 = -\frac{3}{2}, \quad r(e_3) = 2\mu_1\mu_2 = \frac{1}{2},$$
$$\rho = 2(\mu_2\mu_3 + \mu_1\mu_3 + \mu_1\mu_2) = -\frac{5}{2}.$$

**3.4.10 Theorem** (Milnor (1976))**.** If the connected 3-dimensional Lie group $G$ is not unimodular, then its Lie algebra has a basis $e_1, e_2, e_3$ such that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3$$

with $[e_2, e_3] = 0$, and such that the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

has trace $\alpha + \delta = 2$. If we exclude the exceptional case where $A$ is the identity matrix, then the determinant $D = \alpha\delta - \beta\gamma$ provides a complete isomorphism invariant for this Lie algebra.

Milnor connects this fact to show that, for non-unimodular cases, at least the principal Ricci curvatures are negative.

### 3.4.11 On Cartan connection

The thing that makes the bi-invariant metrics nice is the fact that $t \mapsto \exp(tu)$ are geodesics. For the left-invariant one, we lose this property. Here, we relax the requirement that the connection be compatible, and build another connection that makes $t \mapsto \exp(tu)$ still geodesics.

**3.4.12 Definition.** A connection $\nabla$ is left-invariant if

$$\nabla_{(L_a)_*X}((L_a)_*Y) = (L_a)_*(\nabla_X Y) \quad \text{for all } a \in G,\ X, Y \in \mathfrak{X}(G).$$

Such a connection is uniquely determined by its value at the identity, namely by a bilinear form $\nabla^{(e)}$ on $T_eG$.

$$(\nabla_X Y)(g) = dL_g(\nabla^{(e)}(X_e, Y_e)).$$

**3.4.13 Proposition** (Definition)**.** Let $\nabla$ be a left-invariant connection. Then the following assertions are equivalent.

(a) $\nabla^{(e)}$ is skew-symmetric.

(b) For any $u \in T_eG$, $t \mapsto \exp(tu)$ are geodesics.

Such a connection is called a Cartan connection.

*Proof.* We recall that $t \mapsto \exp(tu)$ is $\gamma_u$, the unique smooth curve such that

$$\gamma_u(0) = e, \quad \gamma_u'(0) = u, \text{ and } X(\gamma_u(t)) = \gamma_u'(t),$$

where $X$ is the left-invariant vector field corresponding to $u$. Then $\nabla_{\gamma_u'(t)}\gamma_u'(t) = (\nabla_X X)(\gamma_u(t)) = \nabla^{(e)}(u,u)$. This equality ensures the equivalence. $\square$

The first example is $(u,v) \mapsto \lambda[u,v]$. It is skew-symmetric, hence induces a Cartan connection with torsion $T(u,v) = (2\lambda - 1)[u,v]$ and curvature $R(u,v)w = \lambda(1-\lambda)[[u,v],w]$.

**3.4.14 Proposition.** For any Lie group $G$ there exists a unique symmetric Cartan connection.

*Proof.* We require it to be symmetric, then at the identity $\nabla^{(e)}(u,v) - \nabla^{(e)}(v,u) = [u,v]$. As $\nabla^{(e)}$ is skew-symmetric, this is equivalent to $\nabla^{(e)}(u,v) = \frac{1}{2}[u,v]$. This extends uniquely, giving us what we need. $\square$

This connection still allows us to have (3.3.3) and $R(u,v)w = \frac{1}{4}[[u,v],w]$.

Clearly, (3.3.3) tells us that if the metric is bi-invariant, the unique symmetric connection is the Levi-Civita connection.

# 4. Representations of Semisimple Lie Algebras

The goal of this part is to present the general approaches to understanding the representations of semisimple complex Lie algebras. We begin with a particular case, the simple Lie algebra $\mathfrak{sl}(n,\mathbb{C})$, and provide some examples. There are many ways to see that $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$ are semisimple.

The long-term goal of this part is not only to study complex simple Lie algebras but also Lie groups and real Lie algebras. Here, we start with the complex case, which we complete in this project. We hope that understanding this case first will make it easier to study the other classes later.

## 4.1 Representations of $\mathfrak{sl}(2,\mathbb{C})$

The structure of $\mathfrak{sl}(2,\mathbb{C})$ is entirely given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = H.$$

We will study representations of $\mathfrak{sl}(2,\mathbb{C})$ by considering the irreducible ones. Also, we focus on finite-dimensional representations. So in what follows, the representations are assumed to be finite-dimensional and irreducible unless stated otherwise.
Let $(\varphi, V)$ be a representation of $\mathfrak{sl}(2,\mathbb{C})$. By Theorem 3.2.8, $\varphi(H)$ is diagonalizable. Then

$$V = \bigoplus_{\alpha \in \operatorname{spec}(H)} V_\alpha.$$

For $v \in V_\alpha$, one can show that

$$\varphi(H)(\varphi(X)(v)) = (\alpha+2)\varphi(X)(v), \quad \text{and} \quad \varphi(H)(\varphi(Y)(v)) = (\alpha-2)\varphi(Y)(v).$$

Then $\varphi(X)$ maps $V_\alpha$ to $V_{\alpha+2}$ and $\varphi(Y)$ maps $V_\alpha$ to $V_{\alpha-2}$.

Irreducibility implies that all elements in $\operatorname{spec}(H)$ are congruent modulo $2$. By finiteness,

$$V = \bigoplus_{k=0}^{k=N} V_{\alpha_0+2k}. \tag{4.1.1}$$

It is still a bit early to be convinced that every step decreasing by $2$ must give successive eigenvalues, but Proposition 4.1.2 ensures that if $\alpha_0$ and $\alpha_0 + 2N$ are eigenvalues, then all intermediate values are also eigenvalues.

**4.1.1 Lemma.** For any $v \in V_{\alpha_0}$,

$$\forall k \in \mathbb{N}^* : \quad \varphi(Y)(\varphi(X)^k(v)) = -k(\alpha_0 + k - 1)\varphi(X)^{k-1}(v). \tag{4.1.2}$$

*Proof by Induction.* For $k = 1$, and assuming $\varphi(Y)(v) = 0$, we have

$$-\alpha_0 v = [\varphi(Y), \varphi(X)](v) = \varphi(Y)\varphi(X)(v).$$

Assume it is true up to $k$,

$$\begin{aligned}\varphi(Y)(\varphi(X)^{k+1}(v)) =&[\varphi(Y),\varphi(X)](\varphi(X)^k(v)) + \varphi(X)\Big(\varphi(Y)(\varphi(X)^k(v))\Big)\\ =&(-\alpha_0-2k)\varphi(X)^k(v) - k(\alpha_0+k-1)\varphi(X)^k(v)\\ =&-(k+1)(\alpha_0+k)\varphi(X)^k(v).\end{aligned}$$

Hence (4.1.2). □

**4.1.2 Proposition.** Let $v \in V_{\alpha_0}$. If $v \neq 0$, $W := \operatorname{span}\Big(\{\varphi(X)^k(v) : \quad k \in \mathbb{N}\}\Big) = V$.

*Proof.* Clearly, it is invariant under the action $X$. By noticing that $\varphi(X)^k$ maps $V_\alpha$ to $V_{\alpha+2k}$, it is also invariant under the action of $H$. Lemma 4.1.1 also shows that it is invariant under the action of $Y$. Hence $W$ is invariant under $\varphi$ and, by irreducibility, $W = V$. □

Now, let $M$ be the largest $m \in \mathbb{N}$ such that $\varphi(X)^m(v) \neq 0$. Then $\varphi(X)^{m+1}(v) = 0$. Applying (4.1.2),

$$0 = \varphi(Y)(\varphi(X)^{m+1}(v)) = -(k+1)(\alpha_0+m)\varphi(X)^m(v) \implies \alpha_0 = -m \in \mathbb{Z}.$$

Since for $l \neq k$, $\varphi(X)^k(v)$ and $\varphi(X)^l(v)$ lie in different eigenspaces, then $(\varphi(X)^j)_{j=0}^{j=m}$ are linearly independent, hence a basis. One can deduce also now that, if $d = \dim(V)$, $m = d-1 = N$ and

$$V = V_{1-d} \oplus V_{3-d} \oplus \cdots \oplus V_{d-3} \oplus V_{d-1} =: V^{(d-1)}. \tag{4.1.3}$$

Hence all representations of $\mathfrak{sl}(2,\mathbb{C})$ are of the form (4.1.3).

**4.1.3 Example.** Let $e_1$ and $e_2$ be the standard basis of $\mathbb{C}^2$. $\mathbb{C}^2 = \langle e_1\rangle \oplus \langle e_2\rangle$. The representation

$$\varphi\left(\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\right)\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} ax+by \\ cx-ay\end{pmatrix},$$

of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$ is checkable irreducible. Then it has the form $V^{(1)} = V_{-1} \oplus V_1$ where $V_1 = \langle e_1\rangle$ and $V_{-1} = \langle e_2\rangle$ because $\varphi(H)(e_1) = e_1$ and $\varphi(X)(e_2) = -e_2$.
More generally, the set of homogeneous polynomials in $\mathbb{C}[x,y]$ of degree $d-1$, identified with $\operatorname{Sym}^{d-1}(\mathbb{C}^2)$, is a vector space of dimension $d$, spanned by $\Big(f_k := x^k y^{n-k}\Big)_{k=0}^{k=d-1}$. The endomorphisms

$$\mathcal{H} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}; \quad \mathcal{X} = x\frac{\partial}{\partial y} \quad \text{and } \mathcal{Y} = y\frac{\partial}{\partial x}$$

verify $[\mathcal{H},\mathcal{X}] = 2\mathcal{X}$, $[\mathcal{H},\mathcal{Y}] = -2\mathcal{Y}$ and $[\mathcal{X},\mathcal{Y}] = \mathcal{H}$. The linear map $X \mapsto \mathcal{X}$, $Y \mapsto \mathcal{Y}$ and $H \mapsto \mathcal{H}$ is a representation of $\mathfrak{sl}(2,\mathbb{C})$. Also $f_k$ are eigenvectors of $\mathcal{H}$; $\mathcal{H}$ is diagonalizable with $\operatorname{spec}(H) = (2k-n)_{k=0}^{k=d-1}$. One can also see that $\mathcal{X}$ maps $V_{2k-n}$ to $V_{2k-n+2}$ and $\mathcal{Y}$ maps $V_{2k-n}$ to $V_{2k-n-2}$.
Hence this representation is irreducible. Indeed, if $W$ is invariant, the restriction of $\mathcal{H}$ on $W$ is diagonalizable, hence contains $f_{k_0}$ for some $k_0 = 0,1,\cdots,d-1$. Acting with $\mathcal{X}$ and $\mathcal{Y}$, we get $W = \operatorname{Sym}^{d-1}(\mathbb{C}^2)$.

**4.1.4 Theorem** (Hall (2013))**.** If $(\varphi, V)$ is a finite-dimensional representation of $\mathfrak{sl}(2;\mathbb{C})$, not necessarily irreducible, the following results hold.

(a) Every eigenvalue of $\varphi(H)$ is an integer. Furthermore, if $v$ is an eigenvector for $\varphi(H)$ with eigenvalue $\lambda$ and $\varphi(X)v = 0$, then $\lambda$ is a non-negative integer.

(b) The operators $\varphi(X)$ and $\varphi(Y)$ are nilpotent.

(c) If we define $S : V \to V$ by

$$S = e^{\varphi(X)}e^{-\varphi(Y)}e^{\varphi(X)},$$

then $S$ satisfies

$$S\varphi(H)S^{-1} = -\varphi(H).$$

(d) If an integer $k$ is an eigenvalue for $\varphi(H)$, so is each of the numbers

$$-|k|, -|k|+2, \ldots, |k|-2, |k|.$$

## 4.2 Representations of $\mathfrak{sl}(3,\mathbb{C})$

The structure of $\mathfrak{sl}(3,\mathbb{C})$ is entirely given by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\begin{array}{lll} [H_1, H_2] = 0, & [H_i, X_i] = 2X_i, & [H_2, X_1] = -X_1, \\ [H_1, X_2] = -X_2, & [H_1, X_3] = X_3, & [Y_1, X_3] = X_2, \\ [H_2, X_3] = X_3, & [H_i, Y_j] = -[H_i, X_j], & [Y_2, X_3] = -X_1, \\ [X_1, X_2] = X_3, & [Y_2, Y_1] = Y_3, & [X_3, Y_3] = H_1 + H_2, \\ [X_i, Y_i] = H_i, & [X_i, X_3] = 0, & [Y_i, Y_3] = 0. \end{array}$$

The same for $\mathfrak{sl}(2,\mathbb{C})$, we study an irreducible and finite-dimensional representation $(\varphi, V)$ of $\mathfrak{sl}(3,\mathbb{C})$. One can notice that $\mathfrak{sl}(3,\mathbb{C})$ has the subalgebras $\mathfrak{g}_1 = \langle H_1, X_1, Y_1\rangle$ and $\mathfrak{g}_2 = \langle H_2, X_2, Y_2\rangle$, which are both isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. The idea here is to use the same method as in 4.1, in an appropriate way. As we did with $\mathfrak{sl}(2,\mathbb{C})$, we want to study the representation of $\mathfrak{sl}(3,\mathbb{C})$ through the eigenvalues of $H_1$ and $H_2$. The role of eigenvalues is replaced here by weights.

**4.2.1 Definition.** A weight is a pair $\alpha = (\alpha_1, \alpha_2)$ of eigenvalues of $\varphi(H_1)$ and $\varphi(H_2)$ which have a common eigenvector. This eigenvector is then called a weight vector associated with $\alpha$. The set of weight vectors associated to $\alpha$, denoted $V_\alpha$, is a subspace called the weight space. The multiplicity of $\alpha$ is the dimension of $V_\alpha$.

Since we work over $\mathbb{C}$, $\varphi(H_1)$ admits an eigenvalue $\alpha_1$ corresponding to an eigenspace $W$. The relation $[H_1, H_2] = 0$ implies that $\varphi(H_1)$ and $\varphi(H_2)$ commute, hence $\varphi(H_2)(W) \subseteq W$. Considering the restriction of $\varphi(H_2)$ to $W$, there exists an eigenvalue $\alpha_2$ of $\varphi(H_2)$ with eigenspace $W' \subseteq W$. So

$\alpha = (\alpha_1, \alpha_2)$ is a weight for $(\varphi, V)$.
This shows that any representation admits at least one weight.
One can also notice that, since the restrictions of $\varphi$ to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic to representations of $\mathfrak{sl}(2,\mathbb{C})$, any weight of $(\varphi, V)$ must be an element of $\mathbb{Z}^2$.
Now we need one more definition to understand the action of $\varphi(X_i)$ and $\varphi(Y_i)$ on each weight space of $(\varphi, V)$.

**4.2.2 Definition.** A root is an element $r = (r_1, r_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that there exists $Z_r \in \mathfrak{sl}(3,\mathbb{C})$ with $[H_i, Z_r] = r_i Z_r$. Such a $Z_r$ is called a root vector associated with $r$.

By computation, one can check that the only roots of $\mathfrak{sl}(3,\mathbb{C})$ are

$$a := (2,-1) \leftrightarrow X_1,\ (-2,1) \leftrightarrow Y_1,\ b := (-1,2) \leftrightarrow X_2,\ (1,-2) \leftrightarrow Y_2,\ (1,1) \leftrightarrow X_3,\ (-1,-1) \leftrightarrow Y_3.$$

**4.2.3 Lemma.** If $r$ is a root associated with root vector $Z_r$, then $\varphi(Z_r)$ maps $W_\alpha$ into $W_{\alpha+r}$.

*Proof.* Let $w \in W_\alpha$. Then

$$\varphi(H_i)(\varphi(Z_r)(w)) = [\varphi(H_i), \varphi(Z_r)](w) + \varphi(Z_r)(\varphi(H_i)(w)) = r_i \varphi(Z_r)(w) + \alpha_i \varphi(Z_r)(w) = (\alpha_i + r_i)\varphi(Z_r)(w).$$

Hence $\varphi(Z_r)(w) \in W_{\alpha+r}$. □

By abuse of language, we say that the spectrum of $H$, $\operatorname{spec}(H)$, is just the set of all weights of $(\varphi, V)$.

We want to obtain something analogous to (4.1.1). One can claim that

$$V = \bigoplus_{\alpha \in \operatorname{spec}(H)} V_\alpha. \tag{4.2.1}$$

Indeed, the direct sum is not $\{0\}$ since the representation admits at least one weight. Clearly, it is invariant under the action of $H_1$ and $H_2$. To see that it is invariant under the remaining basis elements, it follows from the fact that they are root vectors and from Lemma 4.2.3.

Now we introduce a partial order on $\operatorname{spec}(H)$. For two weights $\alpha$ and $\beta$, we say $\alpha \leq \beta$ if there exist non-negative integers $p, q$ such that $\beta - \alpha = pa + qb$.

A lowest weight is a meet[1]

One can show that it exists here (by finiteness and irreducibility). It is also an element of $\operatorname{spec}(H)$. Order theory ensures that the meet $\varepsilon$ always exists, and in our case it belongs to $\operatorname{spec}(H)$.

Let us take a moment to see what we can quickly deduce from Lemma 4.2.3. First, since $X_3$ and $Y_3$ are root vectors associated with $a+b$ and $-a-b$, if $\alpha$ is a weight, then each of $(\alpha + ka + kb)_{k\in\mathbb{Z}}$ is either a weight or zero. The same reasoning, using the fact that $X_1$, $X_2$, $Y_1$, and $Y_2$ are root vectors, shows that if $\alpha$ is a weight, then each $(\alpha + ka + lb)_{k,l\in\mathbb{Z}}$ is either a weight or zero. Using irreducibility and finiteness, we obtain

$$\operatorname{spec}(H) \subseteq \{\varepsilon + ka + lb : \quad k = 0, 1, \cdots, N;\ l = 0, 1, \cdots, M\}. \tag{4.2.2}$$

The goal now is to show that $\varepsilon$ is indeed a weight, that it is the lowest weight, and that the inclusion (4.2.2) is in fact close to an equality. Let $\bar{\varepsilon} = \varepsilon + ma$, where $m$ is the minimal non-negative integer such that $\varepsilon + ma \in \operatorname{spec}(H)$. Such $m$ exists because $\varepsilon$ is the meet. There exists a nonzero vector $v$ in $V_{\bar{\varepsilon}}$. Now, by minimality of $m$, we have $\varphi(Y_1)(v) = 0$. By definition of $\varepsilon$, we also have $\varphi(Y_2)(v) = 0$. Since $[Y_1, Y_2] = -Y_3$, we also obtain $\varphi(Y_3)(v) = 0$.

[1] that is, the greatest among the lower bounds (infimum).

**4.2.4 Proposition.** For $v \in V_{\bar\varepsilon} \setminus \{0\}$,

$$\operatorname{span}\left(\{\varphi(X_{i_1})\varphi(X_{i_2})\cdots\varphi(X_{i_k})(v) : k \in \mathbb{N},\ 1 \le i_j \le 3\}\right) = V.$$

*Proof.* Let $W$ be the span above. Clearly, it is invariant under the action of $X_i$. To see that it is invariant under the action of $Y_i$ and $H_i$, we proceed by induction on $k$. For $k = 0$, there is nothing to prove for the $H_i$.

For $Y_i$, we have seen above that $\varphi(Y_i)(v) = 0$. The induction follows from the identity

$$\varphi(Y_i)(\varphi(X_{i_1})\varphi(X_{i_2})\cdots\varphi(X_{i_k})) = [\varphi(Y_i),\varphi(X_{i_1})](\varphi(X_{i_2})\cdots\varphi(X_{i_k}))+\varphi(X_{i_1})\varphi(Y_i)(\varphi(X_{i_2})\cdots\varphi(X_{i_k})).$$

Each term in the sum reduces to the case $k-1$. Thus, $W$ is a subrepresentation of $(\varphi, V)$. □

From this fact, one can also conclude that $V_{\bar\varepsilon}$ has dimension one because the roots associated with the root vectors $X_i$ increase at least one coordinate. Hence $V_{\bar\varepsilon}$ cannot contain $\varphi(X_{i_1})\varphi(X_{i_2})\cdots\varphi(X_{i_k})(v)$ unless $k = 0$.
Another important consequence of this fact is that any element in $V$ has weight of the form $\bar\varepsilon + pa + qb$ with $p, q \ge 0$. This implies directly that $\bar\varepsilon = \varepsilon = (\varepsilon_1, \varepsilon_2)$. Hence the lowest weight always exists.

Using Lemma 4.1.1, $-\varepsilon_1$ and $-\varepsilon_2$ are the largest positive integers such that $\varphi(X_1)^{\varepsilon_1}(v) \neq 0$ and $\varphi(X_2)^{\varepsilon_2}(v) \neq 0$.
By the same reasoning, we can also prove that the highest weight always exists.

**4.2.5 Proposition.** If $\varepsilon$ is the lowest weight and $\mu$ is the highest weight, then

$$\mu_1 = -\varepsilon_2 \quad \text{and} \quad \mu_2 = -\varepsilon_1.$$

*Proof.* Assume $\varepsilon$ is the lowest weight. First, let us show that $\bar\mu = (-\varepsilon_2, -\varepsilon_1)$ is an upper bound of $\operatorname{spec}(H)$. It suffices to notice that for any $p, q \ge 0$ with $p > -\varepsilon_1 - \varepsilon_2$ or $q > -\varepsilon_1 - \varepsilon_2$, the weight $\varepsilon + pa + qb$ cannot occur[2].

Now, if $\alpha$ is a weight, then there exist non-negative integers $p, q$ such that $\alpha = \varepsilon + pa + qb$. Solving $\bar\mu = \alpha + p'a + q'b$, we obtain $p' = -\varepsilon_1 - \varepsilon_2 - p$ and $q' = -\varepsilon_1 - \varepsilon_2 - q$, which are non-negative. Hence $\bar\mu$ is an upper bound.

For a simpler argument, one can use the join (supremum) and expect a symmetry in the weight diagram between the lowest weight $\varepsilon$ and the highest weight $\mu$. So $\mu_1 + \mu_2 = -\varepsilon_1 - \varepsilon_2$. But since $\mu \le \bar\mu$, there exist non-negative $p, q$ such that

$$\begin{aligned}(-\varepsilon_2, -\varepsilon_1) =&(\mu_1,\mu_2) + p(2,-1) + q(-1,2)\\ \implies \mu_1 + \mu_2 =& -\varepsilon_1 - \varepsilon_2 = \mu_1 + 2p - q + \mu_2 - p + 2q\\ \implies p = -q \implies p = q = 0.&\end{aligned}$$

Hence $\mu = \bar\mu$. □

We can rewrite (4.2.2) now

$$\operatorname{spec}(H) \subseteq \{\alpha : \ \varepsilon \le \alpha \le \mu\} = \{\varepsilon + ka + lb = \mu - k'a - l'b : \ k, k', l, l' = 0, 1, \cdots, \mu_1 + \mu_2\}. \tag{4.2.3}$$

[2] Either by a direct computation, or by using the $\mathfrak{sl}(2,\mathbb{C})$ subrepresentation structure in a more conceptual way.

The next task is to show that the diagram of weight is full.

So far, each $k = 0, \cdots, -\varepsilon_1$, $\varphi(X_1)^k(v) \neq 0$ and $\varphi(Y_2)^k(v) \neq 0$; for each $l = 0, \cdots, -\varepsilon_2$, $\varphi(X_2)^l(v) \neq 0$ and $\varphi(Y_1)^l(v) \neq 0$. Also $\varphi(X_1)^{-\varepsilon_1+1}(v) = \varphi(X_2)^{-\varepsilon_2+1}(v) = \varphi(Y_1)^{-\varepsilon_2+1}(v) = \varphi(Y_2)^{-\varepsilon_1+1}(v) = 0$

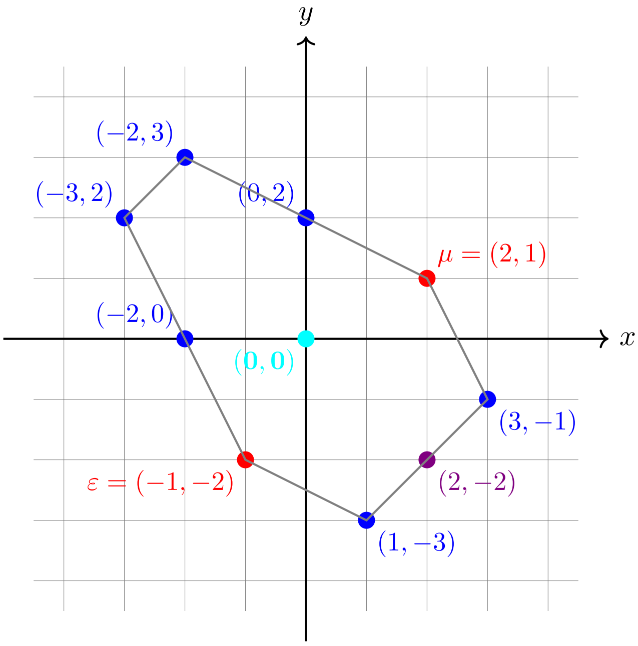


Figure 4.1: Partial weight diagram with lowest weight $\varepsilon = (-1, -2)$.

So far we know that all points in blue (on the edge) of the hexagon in figure 4.1 are weights, in $\mathrm{spec}(H)$. But the point $(2, -2)$, in purple, is not sure yet. One can proceed from above.
I claim that $(-1, 1)$ is also a weight. Why? Because to reach $(-2, 3)$ we have to pass through it, otherwise the path is of the form $X_3X_1X_1(v)$. But saying $(-1, 1) \notin \mathrm{spec}(H)$ implies that the only possible path is $X_3X_1X_1(v) \neq 0$, in particular $X_3X_1(v) = 0$. But this is a contradiction because $[X_1, X_3] = 0$, hence $X_3X_1X_1(v) = X_1(X_3X_1(v)) = 0$.
We can use the same reasoning to complete from top to bottom. So the diagram is full.

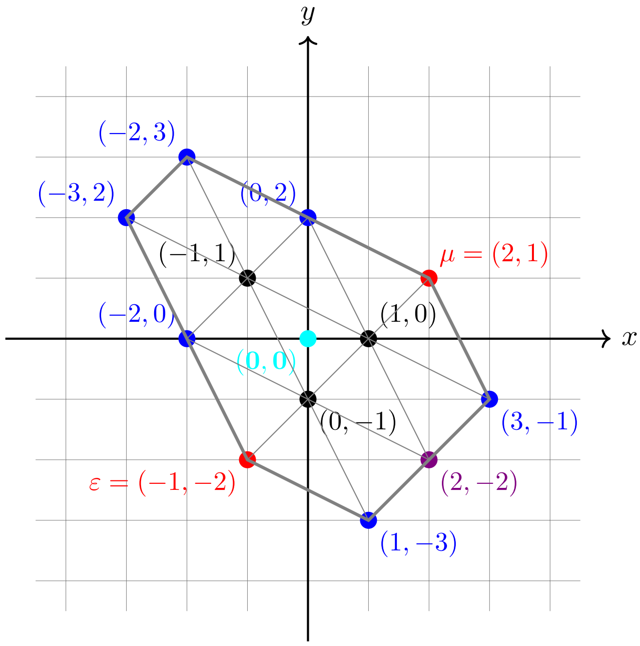


Figure 4.2: Full weight diagram with lowest weight $\varepsilon = (-1, -2)$.

*Sketch of formal (general case) proof to show that the weight diagram is full.*

(i) Notice that any edges (points) in the four segments $[\varepsilon, \varepsilon - \varepsilon_2 b], [\varepsilon, \varepsilon - \varepsilon_1 a], [\mu, \mu + \varepsilon_2 a]$ and $[\mu, \mu + \varepsilon_1 b]$ are all weights.

(ii) Notice that each of the points in the segment $[\varepsilon - \varepsilon_2 b, \mu + \varepsilon_2 a]$ is a weight, using the fact that $X_3$ commutes with $X_k$ and Proposition 4.2.4.

(iii) Repeat the same reasoning recursively for the next oblique segment until the segment $[\varepsilon - \varepsilon_1 a, \mu + \varepsilon_1 b]$.

□

**4.2.6 Proposition.** Two irreducible representations having the same lowest weight are isomorphic.

*Proof.* Very similar to the one in Hall (2013), Proposition 6.15. □

**4.2.7 Example.** First, the trivial example is $\varepsilon = (0, 0)$, that is $\varphi = 0$.
The next standard one is the one of lowest weight $(0, -1)$. That is $\varphi_1$ is just the inclusion from $\mathfrak{sl}(3,\mathbb{C})$ to $\mathfrak{gl}(V)$ for $V = \mathbb{C}^3$. We can check that it is irreducible and $e_1$, $e_2$, $e_3$ have weights respectively $(1, 0), (-1, 1), (0, -1)$.
Now, one can see that the one with lowest weight $(-1, 0)$ is the dual of the standard one above; it can be thought of as just $\varphi_2(Z) = -Z^T$ for $V = \mathbb{C}^3$. $e_1$, $e_2$, $e_3$ have weights respectively $(-1, 0), (1, -1), (0, 1)$.
By uniqueness, for both examples, it is enough to check that they are irreducible and that they verify the properties that we have studied so far.

**4.2.8 Proposition.** For $\varepsilon_1, \varepsilon_2$ non-positive integers, there is an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ that has lowest weight $\varepsilon = (\varepsilon_1, \varepsilon_2)$.

We will discuss the general case in Section 4.3.10.

There are many interesting questions that we should ask about the properties of the unique irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with lowest weight $\varepsilon$, such as the dimension and the multiplicity of each weight.

**4.2.9 Theorem.** Let $V$ be the irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with lowest weight $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Then its dimension is given by

$$\dim V = -\frac{1}{2}(\varepsilon_2 - 1)(\varepsilon_1 - 1)(\varepsilon_1 + \varepsilon_2 - 2).$$

The proof of this formula can be found in Hall (2013). The multiplicity of each weight is already known by Kostant's multiplicity formula, seen in Chapter 10, Section 6, Theorem 10.29 in the same book.

## 4.3 Representations of complex semisimple Lie algebra

We have seen from $\mathfrak{sl}(2,\mathbb{C})$ that the work revolves around the diagonalizable matrix $H$. For $\mathfrak{sl}(3,\mathbb{C})$, we deal with the vectors $H_1, H_2$ and use the fact that they are diagonalizable to conclude that $V$ is a sum of weight spaces. First, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra where $\mathrm{ad}(H)$ is diagonalizable for each $H \in \mathfrak{h}$. Such a subalgebra exists and is unique up to isomorphism. The general study of representations of a Lie algebra will revolve around this Cartan subalgebra $\mathfrak{h}$.
Here, by root $\mathrm{r}$, we mean a non-trivial element of the dual of $\mathfrak{h}$, $\mathfrak{h}^*$ that has a non-trivial root space

$$\mathfrak{g}_{\mathrm{r}} := \{X \in \mathfrak{g} : [H, X] = \mathrm{r}(H)X \text{ for all } H \in \mathfrak{h}\}.$$

As $\mathfrak{g}$ is the sum of all common eigenspaces of $\mathrm{ad}(H)$, $H \in \mathfrak{h}$, we have $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\mathrm{r}\in\Delta} \mathfrak{g}_{\mathrm{r}}$ where $\Delta$ is the set of roots. Combining this with the fact that $\mathfrak{g}$ is semisimple and the Killing form $B|_{\mathfrak{h}}$ is non-degenerate, we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by $H_{\mathrm{r}} \in \mathfrak{h} \mapsto B(H_{\mathrm{r}}, -)$[3]. We will write $\langle \mathrm{r}, \mathrm{s}\rangle = \langle \tilde{H}_{\mathrm{r}}, \tilde{H}_{\mathrm{s}}\rangle = \mathrm{r}(\tilde{H}_{\mathrm{s}}) = \mathrm{s}(\tilde{H}_{\mathrm{r}})$.

By computation, one can easily see that $[\mathfrak{g}_{\mathrm{r}}, \mathfrak{g}_{\mathrm{s}}] \subseteq \mathfrak{g}_{\mathrm{r}+\mathrm{s}}$ (here $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}_t = (0)$ if $t \notin \Delta$). Also, one fact is that each $\mathfrak{g}_r$ is a one-dimensional subspace.
By definition, the root lattice $\Lambda_{\mathfrak{g}}$ is the lattice in $\mathfrak{h}^*$ generated by the roots.

An $\mathrm{r}$-string containing $\mathrm{s}$ is the set of all roots (including $0$) of the form $\mathrm{s} + i\,\mathrm{r}$ for some $i \in \mathbb{Z}$.

**4.3.1 Proposition.** Let $\mathrm{r}$ be in $\Delta$, and let $\mathrm{s}$ be in $\Delta \cup \{0\}$.

(a) The $\mathrm{r}$-string containing $\mathrm{s}$ has the form $\mathrm{s} + i\,\mathrm{r}$ for $-p \le i \le q$ with $p \ge 0$ and $q \ge 0$. There are no gaps. Furthermore,

$$p - q = \frac{2\langle \mathrm{s}, \mathrm{r}\rangle}{\langle \mathrm{r}, \mathrm{r}\rangle}, \text{ and } \frac{2\langle \mathrm{s}, \mathrm{r}\rangle}{\langle \mathrm{r}, \mathrm{r}\rangle} \in \mathbb{Z}.$$

(b) If $\mathrm{s} + i\,\mathrm{r}$ is never $0$, define $\mathfrak{sl}_{\mathrm{r}}$ to be the isomorphic copy of $\mathfrak{sl}(2,\mathbb{C})$ spanned by some $H'_{\mathrm{r}}$, $E'_{\mathrm{r}}$, and $E'_{-\mathrm{r}}$, and let

$$\mathfrak{g}' = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\mathrm{s}+i\,\mathrm{r}}.$$

Then the representation of $\mathfrak{sl}_{\mathrm{r}}$ on $\mathfrak{g}'$ by $\mathrm{ad}$ is irreducible.

This implies also that if $\mathrm{s} = c\,\mathrm{r}$, then $c \in \{-1, 0, 1\}$.

The goal of this section is not to provide all detailed proofs but to describe how the representations look like. The proof can be found in Knapp and Knapp (1996). The key point is that there is a Lie

[3] This is injective, hence an isomorphism since $\mathfrak{h}$ and $\mathfrak{h}^*$ have the same (finite) dimension

subalgebra that acts like $\mathfrak{sl}(2,\mathbb{C})$ and we will use this fact to understand its representations. So we choose or renormalize $H_{\mathrm{r}}$ if needed to get a good basis. Hall gives a more precise description.

**4.3.2 Proposition** (Hall (2013)). For each root $\mathrm{r}$, we can find linearly independent elements $X_{\mathrm{r}}$ in $\mathfrak{g}_{\mathrm{r}}$, $X_{-\mathrm{r}}$ in $\mathfrak{g}_{-\mathrm{r}}$, and $H_{\mathrm{r}}$ in $\mathfrak{h}$ such that $H_{\mathrm{r}}$ is a multiple of $\mathrm{r}$ and such that

$$[H_{\mathrm{r}}, X_{\mathrm{r}}] = 2X_{\mathrm{r}}, \quad [H_{\mathrm{r}}, X_{-\mathrm{r}}] = -2X_{-\mathrm{r}}, \quad [X_{\mathrm{r}}, X_{-\mathrm{r}}] = H_{\mathrm{r}}. \tag{4.3.1}$$

We note that this $H_{\mathrm{r}}$ is not exactly the $\tilde{H}_{\mathrm{r}}$ coming from the non-degenerate Killing form $B$, but the normalized one $\frac{2}{\langle \mathrm{r},\mathrm{r}\rangle}\tilde{H}_{\mathrm{r}}$.

**4.3.3 Definition.** $H_0 \in \mathfrak{h}$ is said to be regular if for any $\mathrm{r} \in \Delta$, $\mathrm{r}(H_0) \neq 0$[4]. We say that $\mathrm{r} \leq \mathrm{s}$ if $\mathrm{r}(H_0) \leq \mathrm{s}(H_0)$, and let $\Delta^+$ be the set of roots $\mathrm{r}$ with $\mathrm{r}(H_0) > 0$. A root $\mathrm{r}$ is fundamental if it is not a sum of two roots in $\Delta^+$. We denote $F = \{f_1, \cdots, f_m\}$ the set of fundamental roots.

**4.3.4 Proposition.** (a) If $f_i, f_j \in F$, then $\langle f_i, f_j\rangle \leq 0$ for $i \neq j$.

(b) The elements of $F$ are linearly independent.

(c) If $\mathrm{r} \in \Delta^+$, then $\mathrm{r} = \sum_{i=1}^m n_i f_i$ with $n_i \geq 0$. *In particular,* $F$ is a basis of $\mathfrak{h}^*_{\mathbb{R}}$.

*Proof.* See Ziller (2010). □

The idea is that all roots are of the form $\mathrm{r}^{(1)}, -\mathrm{r}^{(1)}, \cdots, \mathrm{r}^{(l)}, -\mathrm{r}^{(l)}$, with root vectors $X_1 := X_{\mathrm{r}^{(1)}}, Y_1 := X_{-\mathrm{r}^{(1)}}, \cdots, X_l := X_{\mathrm{r}^{(l)}}, Y_l := X_{-\mathrm{r}^{(l)}}$, and $H_{\mathrm{r}^{(i)}} := H_i$, such that $\{X_i, Y_i, H_i\}$ satisfies (4.3.1). This construction is the main tool we will use; it exists, and the detailed proof is in Hall (2013), Knapp and Knapp (1996). From another point of view, the $\left(\mathrm{r}^{(\sigma(i))}\right)_{i=1}^m$[5] are the fundamental roots $(f_i)_{i=1}^m$, obtained by choosing $m$ positive roots that are not sums of other roots. One can also show that the $(H_i)_{i=1}^m$ span $\mathfrak{h}$. Also, $\mathrm{rk}(\mathfrak{g}) := \dim(\mathfrak{h}) = m$.

Another important generalisation is the notion of weight of a representation $(\varphi, V)$. Again, we assume it is irreducible and $V$ is finite dimensional. An element $\mu \in \mathfrak{h}^*$ is a weight if it has a non-trivial weight space

$$V_\mu := \{v \in V : H \cdot v = \varphi(H)(v) = \mu(H)v \text{ for all } H \in \mathfrak{h}\}.$$

Recall that for each $i = 1, \cdots, m$, $\{X_i, Y_i, H_i\}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Then $\mu(H_i) \in \mathbb{Z}$. Since the $(H_i)$ span $\mathfrak{h}$, we encode the information of $\mathrm{spec}(H)$ in

$$\mathcal{L}(\mathfrak{g}) := \{(\mu(H_1), \cdots, \mu(H_m)) : \quad \mu \in \mathrm{spec}(H)\} \subseteq \mathbb{Z}^m.$$

The clarification is that $\mathcal{L}(\mathfrak{g})$ is what we called $\mathrm{spec}(H)$ in the $\mathfrak{sl}(3,\mathbb{C})$ case. So an element of $\mathcal{L}(\mathfrak{g})$ is an $m$-tuple $(\mu_i)$ such that there exists a nonzero vector in $V$, simultaneously an eigenvector of $\varphi(H_i)$ associated to eigenvalues $\mu_i$. This $\mathcal{L}(\mathfrak{g})$ can be viewed exactly as $\mathrm{spec}(H)$, since a functional is determined by its values on the basis. So we may identify them in what follows.

The same applies to the roots $\Delta$, which can be identified with a subset of $\mathbb{Z}^m$ by considering their values on the $H_i$.

Now we try to repeat the steps from the $\mathfrak{sl}(3,\mathbb{C})$ case with suitable generalisations and small adjustments if needed, keeping in mind that we already treated the problem without explicitly mentioning the Cartan

[4] Such an element always exists because $\Delta$ is finite (we fix one), and we can choose it so that $\alpha(H_0) \in \mathbb{R}$ for each $\alpha \in \Delta$.

[5] $\sigma \in S_n$ for some $m \leq l$.

subalgebra $\mathfrak{h}$, although it was implicitly involved since we worked with $H_1$ and $H_2$ that span $\mathfrak{h}$. Now, by the same computation, Lemma 4.2.3 is still true.

Then we again obtain (4.2.1), since $\mathfrak{g}$ is spanned by $\mathfrak{h}$ and the root vectors.
Usually one works with the partial order defined in Definition 4.3.3, but here we want to proceed in the same way as in the $\mathfrak{sl}(3,\mathbb{C})$ case. So we define, for weights $\alpha, \beta \in \operatorname{spec}(H)$, that $\alpha \preccurlyeq \beta$ means $\beta - \alpha = \sum_{i=1}^{m} n_i f_i$ with $n_i \geq 0$ and $f_i \in F$.

Here, the meet $\varepsilon$ of $\operatorname{spec}(H)$ exists and we claim that it is also the lowest weight.

Let $k = 1, \cdots, m$ and let $m_k$ be the minimal non-negative integer such that $\overline{\varepsilon} = \varepsilon + m_k f_k$ is a weight. Then there exists a nonzero vector $v \in V_{\overline{\varepsilon}}$. The minimality of $m_k$ implies $Y_k \cdot v = 0$. The definition of $\varepsilon$ also implies $Y_i \cdot v = 0$ for each $i = 1, \cdots, m,\ i \neq k$. Since every positive root is a sum of simple roots, any other $Y_j$, $j > m$, can be expressed as brackets of $X_i$, $i = 1, \cdots, m$. Hence for them also $Y_j \cdot v = 0$.

**4.3.5 Proposition.** For $v \in V_{\overline{\varepsilon}} \setminus \{0\}$, $\operatorname{span}\left(\{\varphi(X_{i_1})\varphi(X_{i_2}) \cdots \varphi(X_{i_k})(v) : k \in \mathbb{N},\ 1 \leq i_j \leq l\}\right) = V$.

The proof is very similar to 4.2.4. As in the $\mathfrak{sl}(3,\mathbb{C})$ case, this gives us that $V_{\overline{\varepsilon}}$ is one dimensional and $m_k = 0, \overline{\varepsilon} = \varepsilon$. Since we took an arbitrary $k$, we can say $\varepsilon$ is a weight, then a lowest weight. Using the identification with $\mathfrak{sl}(2,\mathbb{C})$ and lemma 4.1.1, viewing $\varepsilon \in \mathcal{L}(\mathfrak{g})$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_m)$ where the components are the largest positive integers such that $\varphi(X_i)^{\varepsilon_i}(v) \neq 0$.

Let's now talk about an important group that acts on set of weights.

**4.3.6 Definition.** For a root $\mathrm{r} \in \Delta^+$, $s_\mathrm{r} : \mathfrak{h} \longrightarrow \mathfrak{h}, H \mapsto H - \frac{2\langle H,\mathrm{r}\rangle}{\langle \mathrm{r},\mathrm{r}\rangle}\tilde{H}_\mathrm{r} = H - \langle H, \mathrm{r}\rangle H_\mathrm{r}$ is a reflection in the hyperplane $\{M \in \mathfrak{h}_\mathbb{R} | \mathrm{r}(M) = 0\}$. The Weyl group $\mathbf{W}(\mathfrak{g})$ is the group generated by all of the $s_\mathrm{r}$, $\mathrm{r} \in \Delta^+$. The Weyl chamber is $\mathbf{W}_c := \{\, \lambda \in \mathfrak{h}_\mathbb{R}^* \mid \lambda(H_i) \geq 0$ for all $i = 1, \ldots, m \,\}$.

**4.3.7 Remark** (Observation)**.** The following facts are tools that can help us better understand the representation of $\mathfrak{g}$. We omit the proofs, which can be found in Fulton and Harris (2013) or Hall (2013).

(i) $s_\mathrm{r}$ induces a reflection on $\mathfrak{h}^*$ and acts on the roots as $s_\mathrm{r}(\mathrm{s}) = \mathrm{s} - \mathrm{s}(H_\mathrm{r})\,\mathrm{r}$ .

(ii) The set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group.

(iii) $s_\mathrm{r}$ preserves $\mathfrak{h}_\mathbb{R}$.

(iv) Every element of $\mathbf{W}(\mathfrak{g})$ is induced by an automorphism of the Lie algebra $\mathfrak{g}$ carrying $\mathfrak{h}$ to itself.

(v) Every vertex of the convex hull of the weights of $V$ must be conjugate to $\mu$ under the Weyl group, if $\mu$ is the highest weight.

(vi) $\mathbf{W}(\mathfrak{g})$ acts simply transitively on $\mathbf{W}_c$.

From point (v), together with proposition 4.3.1, part (a), we obtain that $\mathcal{L}(\mathfrak{g})$ is of the form $\alpha = \mu + \mathrm{r}$ for some $\mathrm{r} \in \Delta^-$ such that $\alpha$ lies in the convex hull of the weights.

**4.3.8 Theorem** (Fulton and Harris (2013))**.** For any $\mu$ in the intersection of the Weyl chamber $\mathbf{W}_c$ associated to the ordering of the roots with the weight lattice $\Lambda_W$[6], there exists a unique irreducible, finite-dimensional representation $\Gamma_\mu$ of $\mathfrak{g}$ with highest weight $\mu$; this gives a bijection between $\mathscr{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of $\Gamma_\mu$ will consist of those elements of the

[6]The set of all functionals $f$ in $\mathfrak{h}^*$ where $f(H_i) \in \mathbb{Z}$ for each $i$.

weight lattice congruent to $\mu$ modulo the root lattice $\Lambda_{\mathfrak{g}}$ and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to $\alpha$ under the Weyl group.

The proof of existence relies on an abstract construction, but we will prove only uniqueness. To do so, let us start with a fact.

**4.3.9 Lemma.** Let $(\varphi, V)$ be a representation of $\mathfrak{g}$, not necessarily irreducible. If $v$ is a highest weight vector and $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$ is the decomposition with respect to the positive and negative root spaces, then the subrepresentation generated by $v$, $U(\mathfrak{g}) \cdot v$, is equal to $U(\mathfrak{g}^-) \cdot v$, which is the smallest subspace containing $v$ and invariant under the $\mathfrak{g}^-$-action.

*Proof.* Let $w = M_1 M_2 \cdots M_k \cdot v$ and prove by induction on $k$ that it lies in $U(\mathfrak{g}^-) \cdot v$. For $k = 1$, there is nothing to prove. Assume the claim for $k \geq 2$. If $M_1 \in \mathfrak{g}^-$, there is nothing to prove. If $M_1 = H \in \mathfrak{h}$, note that $[H, A_1 A_2 \cdots A_l] = (\theta_1(H) + \cdots + \theta_\ell(H))\, A_1 A_2 \cdots A_\ell$ if $[H, A_j] = \theta_j(H)$. It follows that $M_1 M_2 \cdots M_k \cdot v = M_2 \cdots M_k H \cdot v + (\theta_2(H) + \cdots + \theta_\ell(H)) M_2 \cdots M_k \cdot v \in U(\mathfrak{g}^-)$ by induction. The case $M_1 \in \mathfrak{g}^+$ is handled similarly with a slightly longer computation. $\square$

*Proof of unicity in theorem 4.3.8.* Let $V$ and $W$ be two irreducible representations with the same highest weight $\mu$, with highest weight vectors $v$ and $w$ respectively. Clearly $(v, w)$ is a highest weight vector of $V \oplus W$. By lemma 4.3.9, the subrepresentation of $V \oplus W$ generated by $(v, w)$, $U = U(\mathfrak{g}) \cdot (v, w)$, is equal to $U(\mathfrak{g}^-) \cdot (v, w)$. The top weight space is one dimensional. Now, if $(0) \neq U'$ is a subrepresentation of $U$, there exists $0 \neq u \in U'$. We can raise it using elements of $\mathfrak{g}^+$ until we obtain a nonzero vector $u'$ such that for all $X \in \mathfrak{g}^+$, $X \cdot u' = 0$. Then $u'$ lies in the highest weight space, hence is a scalar multiple of $(v, w)$. Thus $U \subset U'$ and $U$ is irreducible. By Schur's lemma[7], it follows that the restriction of the projections on $U$ are isomorphisms. $\square$

The fundamental weights $\omega_1, \ldots, \omega_m \in \mathfrak{h}^*$ are defined by $\omega_i(H_j) = \delta_{ij}$ for $1 \leq i, j \leq m$. They are all in $\mathbf{W}_c$, and any weight in $\mathbf{W}_c$ can be written uniquely as $\sum_i n_i \omega_i$, $n_i \in \mathbb{N}$.
But now, let us try to give an example to see what we did.

### 4.3.10 Representation of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

It is easy to see that the Cartan subalgebra is $\mathfrak{h} = \{H = diag(h_1, \cdots, h_n) : \quad \sum h_i = 0\}$. Indeed, it is abelian and $\mathrm{ad}(H)$ maps $E_{ij}$ to $(h_i - h_j) E_{ij}$, where $E_{ij}$ denotes the matrix with a $1$ in the $i$-th row and $j$-th column and zeros elsewhere, hence $\mathrm{ad}(H)$ is diagonalizable. To see that $\mathfrak{h}$ is maximal abelian, if $Y \in \mathfrak{sl}(n, \mathbb{C})$ commutes with $\mathfrak{h}$, by computation it must be a diagonal matrix in $\mathfrak{h}$.

We define $\mathrm{r}^{(ij)} : \mathfrak{h} \longrightarrow \mathbb{C}$, $H \mapsto (h_i - h_j)$. One can check that they are roots if $i \neq j$, with associated root spaces $g_{ij} = \langle E_{ij} \rangle$. Also $-\mathrm{r}^{(ij)} = \mathrm{r}^{(ji)}$. We can now think of $X_{\mathrm{r}^{(ij)}} = E_{ij}$. We compute that $B(X, Y) = 2n \operatorname{Tr}(XY)$. Then the identification of $\mathrm{r}^{(ij)}$ in $\mathfrak{h}$, via $B$, is

$$\tilde{H_{ij}} = \frac{1}{2n}(E_{ii} - E_{jj}).$$

One can also show that for each $i \leq n-1$, $\mathrm{r}^i := \mathrm{r}^{(i,i+1)}$ is a fundamental root and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{jj} = 2n\tilde{H_{ij}} := H_i$. So $(H_i)_{i=1}^{i=n-1}$ span $\mathfrak{h}$. The $\langle X_i := X_{\mathrm{r}^{(i,i+1)}}, Y_i := X_{\mathrm{r}^{(i+1,i)}}, H_i \rangle$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Let $(\pi, \mathbb{C}^n)$ be the standard representation, which is clearly irreducible and has weights

$$\mathcal{L}(\mathfrak{g}) = \Big\{(1, 0, \ldots, 0),\ (-1, 1, 0, \ldots, 0),\ (0, -1, 1, 0, \ldots, 0),\ \ldots,\ (0, \ldots, 0, -1, 1),\ (0, \ldots, 0, -1)\Big\},$$

[7] A morphism between irreducible representations is either zero or an isomorphism.

whose lowest is $(0, \dots, 0, -1)$ and highest is $(1, 0, \dots, 0)$.
Next, we claim that $\wedge^k \mathbb{C}^n$ is an irreducible representation of highest weight $\mu^{(k)} = (0, , \cdots, 0, 1, 0, \cdots, 0)$ (1 only in the $k$-th component, zeros otherwise). The basis of $\bigwedge^k \mathbb{C}^n$ is $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}$. By Leibniz rules, each of them has weight $L_{i_1} + \cdots + L_{i_k}$, if $L_j : H \mapsto h_j$. The highest among them is $\mu^{(k)} = L_1 + \cdots + L_k$[8] because $\mu_k - (L_{i_1} + \cdots + L_{i_k}) = \sum_i x_j \, \mathrm{r}^{(j,j+1)}$, for some $(x_j)$. One can show that it is irreducible. Indeed, let $W$ be invariant. Noticing $[E_{a,b}, E_{c,d}] = \delta_{b,c} E_{a,d} - \delta_{a,d} E_{c,b}$, $\mathfrak{b}^+ = \mathrm{spam}(E_{ij} : i < j)$ is a Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$, then we can apply Lie's theorem 3.2.5 for $\mathfrak{b}^+ \subseteq \mathfrak{gl}(W)$ to obtain a nonzero vector $w \in W$, a simultaneous eigenvector of each $E_{ij} \in \mathfrak{b}^+$. As they are nilpotent, then $E_{ij} \cdot w = 0$. Then $w$ is a highest weight vector. We can check that there is no $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with weight $\mu^{(k)}$ other than $v_k = e_1 \wedge e_2 \wedge \cdots \wedge e_k$ (by the fact that the only linear relation among the $L_i$ restricted to $\mathfrak{h}$ is $\sum_{i=1}^{i=n} L_i = 0$). It follows $w = \lambda v_k$, $\lambda \in \mathbb{C}$. That means $v_k \in W$. To conclude $W = \wedge^k \mathbb{C}^n$, let us prove that we can reach any $e_{i_1} \wedge \cdots \wedge e_{i_k}$ using only $v_k$ and the action of $\mathfrak{sl}(n, \mathbb{C})$. It suffices to see $E_{i_k,k} \cdot v_0 = e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{i_k}$, $E_{i_{k-1},k-1} \cdot (E_{i_k,k} \cdot v_0) = e_1 \wedge \cdots \wedge e_{k-2} \wedge e_{i_{k-1}} \wedge e_{i_k}$. Again and again[9] until $E_{i_1,1} \cdots E_{i_k,k} \cdot v_0 = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$.

**4.3.11 Theorem.** For any $\mu \in \mathbb{N}^{n-1}$, there is a unique irreducible representation of highest weight $\mu$,

$$\Gamma_\mu \subseteq W(\mu) = \mathrm{Sym}^{\mu_1}(V) \otimes \mathrm{Sym}^{\mu_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{\mu_{n-1}}(\wedge^{n-1} V).$$

*Proof.* First, highest weights add under tensor products and scale under symmetric powers. Then $W(\mu)$ has highest weight $\mu$. Let $v$ be a weight vector of $\mu$. We claim that if $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, $U(\mathfrak{g}) \cdot v$[10] does the job. To see it we just need to prove the Poincaré–Birkhoff–Witt (PBW) theorem[11], for $\mathfrak{sl}(n, \mathbb{C})$ case: any element $X \in U(\mathfrak{sl}(n, \mathbb{C}))$ can be written as a finite sum, $X = \sum_j x_-^{(j)} x_0^{(j)} x_+^{(j)}$, with $x_-^{(j)} \in U(\mathfrak{b}^-)$, $x_0^{(j)} \in U(\mathfrak{h})$, $x_+^{(j)} \in U(\mathfrak{b}^+)$. If $v$ is a highest weight vector, then each $x_+^{(j)}$ kills $v$, and each $x_0^{(j)}$ acts by a scalar; then $X \cdot v = \sum_j c_j x_-^{(j)} \cdot v$, $c_j \in \mathbb{C}$. So $U(\mathfrak{sl}(n, \mathbb{C})) \cdot v = U(\mathfrak{b}^-) \cdot v$, which is irreducible. Indeed, if $T$ is an invariant subspace of $U(\mathfrak{b}^-) \cdot v$ containing a nonzero vector $u$, we can act successively by elements of $\mathfrak{b}^+$ to reach some $\bar{u} \neq 0$ such that for all $X \in \mathfrak{b}^+$, $X(\bar{u}) = 0$. Let $\lambda$ be the weight of $\bar{u}$ and $\mu - \lambda = \sum_{i=1}^{i=n-1} c_i \, \mathrm{r}^i$, for some $c_i \in \mathbb{N}$; viewing them in each $i$-th copy of $\mathfrak{sl}(2, \mathbb{C})$, $\langle E_{i,i+1}, H_i, E_{i+1,i} \rangle$ inside $\mathfrak{sl}(n, \mathbb{C})$, each $c_i$ has to be zero. Then $\bar{u}$ and $v$ have the same weight and must be proportional, since the highest weight space of $U(\mathfrak{b}^-) \cdot v$ is one dimensional. So $T$ contains $U(\mathfrak{b}^-) \cdot v$. □

The theorem 4.1.4 requires $\mu$ in the Weyl chamber $\mathbf{W}_c$ to be considered as a highest weight for some irreducible representation. For $\mathfrak{sl}(n, \mathbb{C})$, it is clearly $\mathbf{W}_c = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$. This is an identification with $\mathbb{N}^{n-1}$; they are the same, just different points of view. Also, $\mathrm{r}^{(ij)} = L_i - L_j$ and after computation $s_{\mathrm{r}^{(ij)}}$ is the transposition $(L_i, L_j)$. So the Weyl group of $\mathfrak{sl}(c, \mathbb{C})$ is $S_n$.

### 4.3.12 Representation of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$

Recall that $\mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{pmatrix} B & S_1 \\ S_2 & -B^T \end{pmatrix} \middle| \, B \in \mathfrak{gl}(n, \mathbb{C}), \; S_i^T = S_i \right\}$. Like we did with $\mathfrak{sl}(n, \mathbb{C})$, one can show $\mathfrak{h} = \{H = \mathrm{diag}(h_1, \dots, h_n, -h_1, \dots, -h_n) \mid h_i \in \mathbb{C}\}$ is a Cartan subalgebra. Let $M_{k,l} = E_{n+k,l} + E_{n+l,k}$, $N_{k,l} = E_{k,n+l} + E_{l,n+k}$, for $k \leq l \leq n$ and $X_{k,l} = E_{k,l} - E_{n+l,n+k}$, for $k \neq l$, $k, l \leq n$. Then $[H, M_{k,l}] = -(h_k + h_l) M_{k,l}$, $[H, N_{k,l}] = (h_k + h_l) N_{k,l}$, $[H, X_{k,l}] = (h_k - h_l) X_{k,l}$.

[8] $L_i : \mathfrak{h} \longrightarrow \mathfrak{h}, H \mapsto h_i$.
[9] We used that $E_{ab} e_b = e_a$, $E_{ab} e_j = 0$ for $j \neq b$, and the Leibniz rule on wedges.
[10] smallest subspace invariant under the action of $\mathfrak{g}$ and containing $v$.
[11] The idea is similar to lemma 4.3.9.

These relations tell us the roots are $\pm\,\mathrm{s}^{(ij)} : \mathfrak{h} \longrightarrow \mathbb{C}$, $H \mapsto \pm(h_i+h_j)$ for $i \leq j \leq n$ and $\mathrm{r}^{(ij)} : \mathfrak{h} \longrightarrow \mathbb{C}$, $H \mapsto (h_i-h_j)$ for $i \neq j \leq n$. The root space $g'_{ij}$ associated to root $\mathrm{s}^{(ij)}$ is spanned by $M_{ij}$, $g''_{ij}$ associated to root $-\,\mathrm{s}^{(ij)}$ is spanned by $N_{ij}$, and $g_{ij}$ associated to root $\mathrm{r}^{(ij)}$ is spanned by $X_{ij}$. We compute that $B(X,Y) = (2n+2)\,\mathrm{Tr}\,(XY)$. Then the identification of $\mathrm{s}^{(ij)}$ and $\mathrm{r}^{(ij)}$ in $\mathfrak{h}$, via $B$, is

$$\tilde{K_{ij}} = \frac{1}{2n+2}(K_i + K_j) \text{ and } \tilde{H_{ij}} = \frac{1}{2n+2}(K_i - K_j), \text{ where } K_i = X_{ii}.$$

Clearly $F = \{\mathrm{r}_1 := \mathrm{r}^{(1,2)}, \cdots, \mathrm{r}_{n-1} := \mathrm{r}^{(n-1,n)}, \mathrm{r}_n = \mathrm{s}^{(n,n)}\}$ is linearly independent and we can generate every root from them; then, by a suitable choice of regular element, $F$ can be taken as fundamental roots.
Now let $X_i := X_{i,i+1}$, $Y_i := X_{i+1,i}$ for $i \leq n-1$ and $X_n := N_{n,n}$, $Y_n := M_{n,n}$. We can check by computation that if $H_i = [X_i, Y_i]$ (it is $(2n+2)\tilde{H}_{ii}$ for $i \leq n-1$ and $(2n+2)\tilde{K}_{ii}$ for $i = n$), then $\langle X_i, Y_i, H_i\rangle$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ and $(H_i)_{i=1}^{i=n-1}$ span $\mathfrak{h}$.

Consider the representation of $\mathfrak{sp}(n,\mathbb{C})$ on $W = \mathbb{C}^{2n}$, $\pi$ by $X \cdot v = Xv$. It is irreducible with weight

$$\mathcal{L}(\mathfrak{g}) = \{\pm w^{(k)} \mid 1 \leq k \leq n\}, \quad w^{(1)} = (1,0,\dots,0),\ w^{(k)} = (0,\dots,0,-1,1,0,\dots,0),\ w^{(n)} = (0,\dots,0,-1,1).$$

The highest weight is now $w^{(1)}$.
Now consider $\pi_k = \wedge^k W$. It is not irreducible for $2 \leq k \leq 2n-2$. Indeed, choosing a symplectic invariant form $\omega \in \wedge^2 W$, one can define $L : \wedge^{k-2} W \longrightarrow \wedge^k W$ by $\alpha \mapsto \omega \wedge \alpha$. We claim that the image of $L$ is invariant under the action and is not zero, hence we obtain a subrepresentation that is neither zero nor all of $\wedge^k W$. By duality, we should expect irreducibility for the cases $k = 2n-1$ and $2n$; for $\mathfrak{sp}(n,\mathbb{C})$, $\wedge^{2n-1} W \cong \wedge^1 W = W$.
This representation has weights $\mathcal{L}(\mathfrak{g}) = \{\pm w^{(i_1)} \pm \cdots \pm w^{(i_l)} \mid i_1 < \cdots < i_l \leq k,\ k-l \text{ even}\}$.
Define $K : \wedge^k W \longrightarrow \wedge^{k-2} W$ by $K(v_1 \wedge \cdots \wedge v_k) := \sum_{1 \leq i<j \leq k} \omega(v_i, v_j)\, v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_k$; $(\omega(Mu,v) + \omega(u,Mv) = 0)$. We claim that $\Gamma_k := \mathrm{Ker}\,K$ is $\mathfrak{g}$-invariant.

Let $x \in \ker K,\ M \in \mathfrak{sp}(n,\mathbb{C})$.

$$\begin{aligned} K(\pi_k(M)\,(v_1\wedge\cdots\wedge v_k)) &= \sum_{p=1}^{k} K(v_1\wedge\cdots\wedge M v_p\wedge\cdots\wedge v_k) \\ &= \sum_{i<j} \omega(v_i,v_j)\,\pi_{k-2}(M)(v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\wedge v_k) \\ &\quad + \sum_{i<j} (\omega(Mv_i,v_j) + \omega(v_i, Mv_j))\, v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\wedge v_k \\ &= \pi_{k-2}(M)\,K(v_1\wedge\cdots\wedge v_k). \end{aligned}$$

Therefore, for $x \in \ker K$, $K(\pi_k(M)x) = \pi_{k-2}(M)\,K(x) = 0 \ \Rightarrow\ \pi_k(M)x \in \ker K$.

Moreover, it is irreducible and one can show it has highest weight $\mu_k(H) = h_1 + \cdots + h_k$ for $k \leq n-1$. So we have now the irreducible representations corresponding to the fundamental weights.

**4.3.13 Corollary.** For any $\mu \in \mathbb{N}^{n-1}$, there is a unique irreducible representation of highest weight $\mu$,

$$\Gamma_\mu \ \subseteq\ W(\mu) = \mathrm{Sym}^{\mu_1}(W)\ \otimes\ \mathrm{Sym}^{\mu_2}(\Gamma_2)\ \otimes\ \cdots\ \otimes\ \mathrm{Sym}^{\mu_n}(\Gamma_n).$$

The proof is similar to what we did with 4.3.11.

# 5. Final Remarks

The Lie algebra of a Lie group encodes a great deal of information about the group itself. In many cases, the classification of Lie algebras leads directly to the classification of Lie groups up to local isomorphism. Although this project did not cover the full classification theory, it is worth noting that complex semisimple Lie algebras are already completely classified via Dynkin diagrams, a remarkably elegant combinatorial tool. Due to time constraints, we have not presented the theory of Dynkin diagrams in detail.

The study of Dynkin diagrams, together with the theory of the Weyl group, provides the complete classification of complex semisimple Lie algebras. These results also shed light on the classification of real semisimple Lie algebras and, consequently, on semisimple Lie groups. In particular, the passage from complex to real forms involves understanding the structure of non-compact real forms, a subject rich in both algebraic and geometric insights.

So this thesis is just a starting point for the study, but we expect to be able to do similar things for real ones and other interesting classes of Lie algebras and Lie groups.

While this thesis focused on the interplay between Lie groups, their Lie algebras, and their representation theory, much remains to be explored. Also, studying the representation theory with plethysm can be more combinatorial, which is also very interesting. The tools developed here—geometric analysis via left-invariant metrics, the study of exponential maps, and the use of representation theory—form a solid foundation for deeper investigations into the classification and structure theory of Lie groups. Knowing the curvatures may help us to understand the Lie groups and vice versa.

Further directions include studying the role of Cartan connections in the classification of homogeneous spaces and symmetric spaces, exploring the relation between curvature and representation theory, and applying these ideas to the theory of real semisimple Lie groups. Also, having Cartan connections, we can study geodesics in terms of algebraic concepts: the exponential map.

# References

Alexandrino, M. M., Bettiol, R. G., et al. *Lie groups and geometric aspects of isometric actions*, volume 82. Springer, 2015.

Do Carmo, M. P. and Flaherty Francis, J. *Riemannian geometry*, volume 2. Springer, 1992.

Fulton, W. and Harris, J. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 2013.

Hall, B. C. Lie groups, lie algebras, and representations. In *Quantum Theory for Mathematicians*, pages 333–366. Springer, 2013.

Helgason, S. *Differential geometry and symmetric spaces*, volume 341. American Mathematical Society, 2024.

Knapp, A. W. and Knapp, A. W. *Lie groups beyond an introduction*, volume 140. Springer, 1996.

Lee, J. M. *Smooth manifolds*. Springer, 2003.

Milnor, J. Curvatures of left invariant metrics on lie groups. *Advances in mathematics*, 21(3):293–329, 1976.

Spivak, M. D. A comprehensive introduction to differential geometry. *(No Title)*, 1970.

Ziller, W. Lie groups, representation theory and symmetric spaces. Lecture notes, University of Pennsylvania, Fall 2010, 2010.