Free Objects in Categories

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Abstract

It is common in many mathematical contexts to have the notion of a “free” structure on a collection of generators. Typically these are in algebraic settings where the free algebra on a set of generators can be understood as the generic algebraic structure containing those generators and which is “free” of any relations or restrictions besides those that are imposed by the axioms of the algebra itself.

In a free group, each element is uniquely represented as a reduced word composed of generators and their inverses. These words can be manipulated using concatenation and cancellation. Free groups arise in various contexts.

In combinatorial group theory, they serve as a model for understanding the structure of groups and their interactions with other algebraic structures. They also arise naturally in topology and geometry, particularly in algebraic topology.

This project will start with an overview of free structures and some of their applications. It will then introduce the category theory required to describe free objects using the language of adjunctions. This categorical insight gives both a unity to the many examples and opens up further examples and applications of free objects.

Keywords: free functors, category, pointfree topology, free object, adjunction, frame, semilattice.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

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1. Introduction

The notion of free object has its roots in various branches of mathematics, where structures like free groups, free lattices, and free algebras emerged in response to the need to understand systems generated in the freest possible way. This idea was formalized and extended to other structures through category theory, a field that unifies and generalizes many mathematical structures and constructions.

Walther Dyck is credited with noting the importance of free groups in combinatorial group theory in (Dyck, 1882), and thereafter the initial explorations of free structures followed in the early 20th century. For example, Jakob Nielsen introduced free bases for free groups around the 1920s, and his work was fundamental to the understanding of free groups. In his paper "On non-commutative rings and their use in group theory" (Nielsen, 1921) he showed that we can construct a set of free generators for the subgroup generated by a set of words in a free group. Simultaneously, Max Dehn studied the fundamental groups of surfaces, introducing methods that are crucial for today's theory of free groups in his work which was later translated in "Transformations of curves on two-sided surfaces" (Dehn, 1987).

In the 1930s and 1940s, lattice theory, largely developed by Garrett Birkhoff and Alfred Tarski, highlighted free constructions within lattices and Boolean algebras. Notably, Philip Whitman provided the first explicit construction of a free lattice on more than 3 generators (Whitman, 1941). The work of Marshall Hall and Philip Hall in the 1950s further deepened our understanding of free algebras, particularly in associative and Lie contexts.

Indeed, in most algebraic settings the question of describing the "free" objects became an important one. Their study is motivated by their ability to model systems in the most general possible way. In many contexts, constructing a free object involves capturing the essential relationships among the generators of a structure without imposing additional constraints. For example, a free group on a set of generators is the most general group containing those generators with no relations other than those necessary to satisfy the group axioms. As such free algebras are fundamental tools for understanding the underlying structures of algebraic systems.

Free objects play a crucial role in various branches of mathematics, not only in algebra but also in topology and logic, for instance. In topology, free spaces and metric spaces are often trivial constructions, but free objects appear in the study of fundamental groups and covering spaces and algebraic topology. In logic, free constructions are essential for understanding model theories and rewriting systems notably in free boolean algebras, and free Heyting algebras.

Pointfree topology is an approach to topology in which the lattice of open sets of a topological space is the central object of study, not the points of the space, and in which we think of spaces as purely algebraic objects. In this approach, free semilattices, free lattices, free frames and free complete Heyting algebras are interesting subjects to explore and understand.

Category theory provides a unifying theory and language in mathematics. We can use this language to extract the essential nature of free objects and arrive at a single definition that applies in many contexts. From this point of view, the free construction amounts to finding the left adjoint of a forgetful functor. Adjoint functors in turn interact nicely with limits and colimits which means that free constructions often assist in the identification and construction of limits and colimits in specific contexts.

We give special attention to the construction of free frames. Johnstone (1982) and Picado and Pultr (2021) both provide free constructions for semilattices, lattices and frames. We follow Johnstone's lead where moving from semilattices to frames, he introduces the concepts of coverage and site.

We extend his definition of maps that "transform covers to joins" to get the morphisms of the category which we establish. We show that the notion of "frame freely generated by a site" can be seen as free object of a site with respect to a functor which we will construct.

We also provide left and right adjoints to the trivial functor from frames to sites.

Outline of the Project

In this project, we will explore the concept of free objects within the framework of category theory. Our investigation is structured as follows.

Chapter 1, the current chapter, provides a brief introduction.

Chapter 2 covers the foundations of category theory. We will begin with the basics, including essential definitions and general concepts such as categories, functors, and natural transformations. We will also delve into specific topics like the category of frames, limits and colimits, and adjunctions.

Chapter 3 is dedicated to understanding free objects through classical constructions. We will examine various examples, including free monoids, free groups, free semilattices, free frames, and free Heyting algebras. Through these examples, we aim to illustrate the construction and significance of free objects in different mathematical contexts. We will also provide some forgetful functors that do not admit free functors.

Chapter 4 touches on what free constructions can provide to mathematics and how one establishes the non-existence of free objects or ways to construct free objects using categorical methods.

2. Category Theory

We start by providing the important concepts in category theory. Most of the definitions, theorems, propositions, and constructions in this chapter are taken from Leinster (2014), Riehl (2017), and Awodey (2010). We give the basic concept in category theory: functors, natural transformations, limits, and andjunctions which we need to define the free constructions.

2.1 Generality

2.1.1 Definition. A **category** $\mathcal{C}$ consists of:

(i) a collection of objects whose each element is called $\mathcal{C}$-object or object of $\mathcal{C}$;

(ii) a collection of morphisms each having an object as its domain and an object as its codomain whose each element is called $\mathcal{C}$-morphism or arrow (morphism) in $\mathcal{C}$;

(iii) a composition $\circ$, defined for pair (g, f) of morphisms if domain of g and codomain of f are the same and assigns to a new morphism $g \circ f$ of domain f'domain and codomain g'codomain;
so that:

(iv) for each object O, there is an identity morphism of O with domain and codomain O, written $\mathbf{1}_O$;

(v) for any objects O, P, Q, R and morphisms f, g, h with domains O, P, Q and codomains P, Q, R:

- $f \circ \mathbf{1}_O = f = \mathbf{1}_P \circ f$;
- $h \circ (g \circ f) = (h \circ g) \circ f$.

2.1.2 Remark. We will write "$A \in \mathcal{C}$" instead of A is an object of $\mathcal{C}$ and "f ***in*** $\mathcal{C}$" instead of f is a morphism of $\mathcal{C}$. We denote $Ob(\mathcal{C})$, the collection of objects of $\mathcal{C}$ (the $\mathcal{C}$-objects) and $Mor(\mathcal{C})$, the collection of morphisms of $\mathcal{C}$ (the $\mathcal{C}$-morphisms). If the morphism f has domain A and codomain B, we will simply write $f : A \to B$ or $A \xrightarrow{f} B$. We can simply write fg for $f \circ g$ and call it "f composed with g". We denote $\mathcal{C}(A, B)$ the collection of all morphisms with domain A and codomain B.

2.1.3 Example. We can easily check that we have a category of sets, $\mathbf{Set}$ where the objects are all sets, the morphisms are the maps, the identity is the identity map of each set and the composition is the map composition. We can imagine similarly the category of semigroups $\mathbf{Sgrp}$, monoids $\mathbf{Mon}$, groups $\mathbf{Grp}$, rings $\mathbf{Rng}$, unitary rings $\mathbf{Ring}$, vector space over $\mathbb{K}$, $\mathbf{Vect}_{\mathbb{K}}$ where the morphisms are semigroups homomorphisms, monoid homomorphisms, group homomorphisms, ring homomorphisms, linear maps. We also have the category of Topological spaces $\mathbf{Top}$ where the morphisms are the continuous maps with the composition as maps.

2.1.4 Definition (Riehl (2017) p8)**.** A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is defined by restricting to a subcollection of objects and subcollection of morphisms subject to the requirements that the subcategory $\mathcal{D}$ contains the domain and codomain of any morphism in $\mathcal{D}$, the identity morphism of any object in $\mathcal{D}$, and the composite of any composable pair of morphisms in $\mathcal{D}$.

2.1.5 Example. $\mathbf{Grp}$ is a subcategory of $\mathbf{Mon}$. $\mathbf{Mon}$ is a subcategory of $\mathbf{Sgrp}$.

2.1.6 Definition. A category $\mathcal{C}$ **is small** if it has a small set of objects and a small set of morphisms. More precisely, $Ob(\mathcal{C})$ and $Mor(\mathcal{C})$ are sets. On the other hand, a category $\mathcal{C}$ **is locally small** if $\mathcal{C}(A, B)$ is a set for any A, B objects of $\mathcal{C}$. In this case, we call it a hom-set, often denoted by $\mathcal{H}om_{\mathcal{C}}(A, B)$.

2.1.7 Example. If we consider the poset $(X, \leq)$ as a category (see Remark 2.2.2), it is small because $Ob(X, \leq) = X$ and $Mor(X, \leq) = \leq$ as a subset of $\mathcal{P}(X)$ are both sets. The category $\mathbf{Set}$ is not small because the collection of all sets, $Ob(\mathbf{Set})$ is not a set; but it is locally small because for each set A and B, $\mathcal{C}(A, B)$ is a set.

2.1.8 Definition. Given a category $\mathcal{C}$, we can define a dual category $\mathcal{C}^{op}$, the **opposite** of $\mathcal{C}$. $\mathcal{C}$ and $\mathcal{C}^{op}$ have the same collection of objects and $\mathcal{C}^{op}(A, B) = \mathcal{C}(B, A)$ for any A, B objects of $\mathcal{C}$. For f^{op} in $\mathcal{C}^{op}(A, B)$ and g^{op} in $\mathcal{C}^{op}(B, C)$ (with representantives in $\mathcal{C}$ are $f \in \mathcal{C}(B, A)$ and $g \in \mathcal{C}(C, B)$), the composition, in $\mathcal{C}^{op}$, $g^{op} f^{op}$ is the morphism of representive $f \circ g$ in $\mathcal{C}(C, A)$.

2.1.9 Example. Let $(X, \leq)$ be a poset. Let's consider $\geq$ the opposite of the relation $\leq$. It means that for any $x, y \in X$: $x \geq y$ if and only if $y \leq x$. We know that such a relation gives us a new poset $(X, \geq)$. Considering $(X, \leq)$ and $(X, \geq)$ being category, we can verify that $(X, \geq)$ is opposite of $(X, \leq)$.

2.1.10 Definition (Monomorphism, Epimorphism, Isomorphism)**.** Let $f : B \longrightarrow C$ *in* $\mathcal{C}$.

(a) f is a **monomorphism** if for every $g, h : A \longrightarrow B$ *in* $\mathcal{C}$, $fg = fh$ implies $g = h$.

(b) f is an **epimorphism** if for every $g, h : C \longrightarrow D$ *in* $\mathcal{C}$, $gf = hf$ implies $g = h$.

(c) f is an **isomorphism** if there is a morphism $g : C \longrightarrow B$ *in* $\mathcal{C}$ such that $gf = \mathbf{1}_B$ and $fg = \mathbf{1}_C$. Such g is necessarily unique, denoted by f^{-1}. In this case, C and B are said to be isomorphic, writen $C \cong B$.

2.1.11 Example. For the categories, $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Top}$, epimorphism is the same as surjective and monomorphism is the same as injective. For $\mathbf{Ring}$, epimorphisms are not always surjective. The inclusion from $\mathbb{Z}$ to $\mathbb{Q}$ is a counter-example.

2.1.12 Proposition. If a morphism $f : B \longrightarrow C$ is an isomorphism, then f is an epimorphism and a monomorphism.

Proof. Let $a, b : A \longrightarrow B$ and $a', b' : C \longrightarrow D$ such that $fa = fb$ and $a'f = b'f$ then $f^{-1}fa = f^{-1}fb$ and $a'ff^{-1} = b'ff^{-1}$. So, $\mathbf{1}_B a = \mathbf{1}_B b$ and $a'\mathbf{1}_C = b'\mathbf{1}_C$. Thus $a = b$ and $a' = b'$. □

2.1.13 Remark. It is not enough for a morphism to be an isomorphism if it is a monomorphism and an epimorphism. For instance, the morphism $x \mapsto x$ from $\{a, b\}$ with discrete topology to $\{a, b\}$ with indiscrete topology is an epimorphism and monomorphism but not isomorphism in the category $\mathbf{Top}$.

2.2 About frames, locales, and Heyting algebra

In this section, we describe the objects that we use frequently in Pointfree Topology. The most important are those that concern frame, localic, and Galois connections found in the excellent book by Picado and Pultr (2011).

2.2.1 Definition. A **poset** is given by a couple $(X, \leq)$ where X is a set and $\leq$ is a partial order relation on X. If we consider the map between two posets which preserves order, we get: a category of poset $\mathbf{Pos}$ whose objects are posets and morphisms are order-preserving maps. Composition is the composition of two maps.

2.2.2 Remark. For a given poset $(X, \leq)$ we can see X as a category where the objects $Ob(X) = X$ and the only morphism $a \to b$ is $a \leq b$ if this is the case but $X(a, b) = \emptyset$ otherwise. The composition is just the transitivity. The identity morphism is ensured by reflexivity, and the associativity is by the uniqueness of morphism from one object to another if it exists. (This category has nothing to do with $\mathbf{Pos}$).

2.2.3 Definition. A **meet-semilattice** is a poset that admits all finite meets, including a top element 1 (here, with bottom 0). We get the category of semilattice $\mathbf{MSLat}^*$ if we consider the morphisms as the structure-preserving maps (semilattice homomorphism). Composition is the composition of two maps.

For the meet-semilattice, not necessarily with bottom 0, we denote it $\mathbf{MSLat}$.

2.2.4 Remark. A meet-semilattice is entirely determined by the meet operation. Indeed, for $(S, \wedge) \in \mathbf{MSLat}$, the order is given by $a \leq b$ if and only if $a \wedge b = a$. The preserving order property also follows.

2.2.5 Definition. A **lattice** is a poset which admits all finite meets and finite joins. We obtain the category of lattices $\mathbf{Lat}$ if we consider the morphisms as the structure-preserving maps (lattice homomorphism). Composition is the composition of two maps.

2.2.6 Definition. A **complete lattice** is a poset which has all joins and all meets. We obtain the category of complete lattice $\mathbf{CLat}$ if we consider the morphisms as the structure-preserving maps.

2.2.7 Definition. A **frame** is a complete lattice L satisfying the distributive law

$$a \wedge (\bigvee_{b \in B} b) = \bigvee_{b \in B} (a \wedge b) \quad \text{for any } a \in L \text{ any } B \subseteq L.$$

We get the category of frames $\mathbf{Frm}$ if we consider the morphisms as the structure-preserving maps (frame homomorphism) in the sense of preserving order, top, finite meets, and arbitrary joins.

The Category of locales, $\mathbf{Loc}$ is the opposite of $\mathbf{Frm}$.

2.2.8 Example. If we have a topological space (X, τ_X), the poset $(\tau_X, \subseteq)$ is a frame. Indeed, the intersection (meet) of finite open sets is an open set and the union (join) of any open sets is an open set. Note that the arbitrary meet is the interior of intersection. Moreover, the distributive law is clear from the property of union and intersection.

2.2.9 Definition. For a two $\mathbf{Pos}$-morphisms $(A, \leq) \underset{g}{\overset{f}{\rightleftarrows}} (B, \leq)$, f is a left adjoint to g (or g right adjoint to f) means

$$\forall a \in A, b \in B : f(a) \leq b \iff a \leq g(b). \tag{2.2.1}$$

2.2.10 Remark. The adjoints of a given map do not always exist but, if it exists, it is unique.

2.2.11 Remark. The left adjoints preserve all joins, and the right adjoints preserve meets. We will prove this and discuss its converse in general category in Theorems 2.7.10 and 2.7.11 but for now, we skip the particular proof. These particular cases are found in the Appendix I.5 of Picado and Pultr (2011).

2.2.12 Remark. Any $\mathbf{Frm}$-morphism $A \xrightarrow{f} B$ admits a right adjoint g, which can be expressed:

$$\forall b \in B : g(b) := \bigvee\{x \in A : f(x) \leq b\}. \tag{2.2.2}$$

Because of the uniqueness of adjoints, we can consider the $\mathbf{Loc}$-morphism (localic map), as a right adjoint to a $\mathbf{Frm}$-morphism.

2.2.13 Definition. Let L be a lattice. A binary operation $\to$ on L is called the Heyting operation(arrow) if for any $a, b, c \in L$,

$$a \leq (b \to c) \iff a \wedge b \leq c.$$

2.2.14 Remark. For any $a \in L$, the preserving order map $(a \to -)$ is the right adjoint to the order-preserving map $(a \wedge -)$. So, the Heyting operation does not always exist, and if it exists, it is unique.

2.2.15 Definition. The Heyting algebra is a bounded lattice with a Heyting operation. A morphism of Heyting algebra is a $\mathbf{Lat}$-morphism which preserves the Heyting operation. We get a category $\mathbf{Hey}$ of Heyting algebra. If in addition, the lattice is complete, we call complete Heyting algebra, and the category corresponding, we will denote $\mathbf{CHey}$.

2.2.16 Theorem. A complete lattice L admits a Heyting arrow if and only if it is a frame.

Proof. From left to right, we use Remarks 2.2.14 and 2.2.11. From right to left, we can check:

$$a \to b = \bigvee\{x \in L : a \wedge x \leq b\}$$

verifies for all $c \in L$: $c \leq a \to b$ if and only if $c \wedge a \leq b$. Indeed, if $c \wedge a \leq b$, $c \in \{x \in L : a \wedge x \leq b\}$. Then $c \leq a \to b$. Finally, if $c \leq a \to b$, $a \wedge c \leq a \wedge \bigvee\{x \in L : a \wedge x \leq b\} = \bigvee\{a \wedge x : x \in L, a \wedge x \leq b\} \leq b$. □

2.2.17 Definition. A subset S of locale L is said to be sublocale of L if it is closed under all meets and absorbs Heyting of elements in the sense that for any $s \in S$ and $x \in L$, $x \to s \in S$.

2.2.18 Proposition (Picado and Pultr (2011))**.** Let L be a locale. A subset K of L is a sublocale of L if and only if K is a locale in the induced order and the embedding map is a localic map.

Proof. See the proposition 2.2 in Picado and Pultr (2011), page 26. □

2.3 Functor and natural transformation

2.3.1 Definition. For categories $\mathcal{D}$ and $\mathcal{C}$, a **functor** $\mathbb{F} : \mathcal{D} \longrightarrow \mathcal{C}$ consists of two suitably related mappings: **the object mapping** $\mathbb{F}$, which assigns to each object O of $\mathcal{D}$ to an object $\mathbb{F}O$ (or $\mathbb{F}(O)$) of $\mathcal{C}$ and **the morphism mapping** also $\mathbb{F}$, which assigns to each morphism f of $\mathcal{D}$ to a morphism $\mathbb{F}f$ (or $\mathbb{F}(f)$) of $\mathcal{C}$ that respects the domains, codomains, the identity, and the composition.

2.3.2 Example. $\mathbf{Lc} : \mathbf{Top} \longrightarrow \mathbf{Loc}$

$$\begin{aligned} \mathbf{Lc}: \quad & \mathbf{Top} \to \mathbf{Loc} \\ & (X,\tau_X) \mapsto \mathbf{Lc}(X,\tau_X) = (\tau_X, \subseteq) \\ (X,\tau_X) \xrightarrow{f} (Y,\tau_Y) & \mapsto (\tau_X,\subseteq) \xrightarrow{\mathbf{Lc}(f)} (\tau_Y,\subseteq) \text{ with } \mathbf{Lc}(f) : U \mapsto Y \setminus \overline{f[X\setminus U]}. \end{aligned} \tag{2.3.1}$$

For each object, it is well-defined by Example 2.2.8. To check why the morphism mapping is well-defined: we know that $f^{-1}[B] \subseteq A$ if and only if $B \subseteq Y \setminus f[X \setminus A]$ for any map $f : X \longrightarrow Y$, subset B of Y and subeset A of X. Then for any continuous maps $f : (X,\tau_X) \longrightarrow (Y,\tau_Y)$ and for any $A \in \tau_X$, $B \in \tau_Y$, $f^{-1}[B] \subseteq A$ if and only if $B \subseteq (Y \setminus f[X \setminus A])^{\circ} = Y \setminus \overline{f[X \setminus A]}$.
Then the inclusion-preserving map, $A \mapsto Y \setminus \overline{f[X \setminus A]}$ is a right adjoint to f^{-1} which we will show at Example 2.3.7 as $\mathbf{Frm}$-morphism. So the mapping $U \mapsto Y \setminus \overline{f[X \setminus U]}$ is a $\mathbf{Loc}$-morphism.

Moreover, it respects domains and codomains by construction. Obviously, it respects identity.

Now, let $f : (X,\tau_X) \longrightarrow (Y,\tau_Y)$ **in** $\mathbf{Top}$, $E \subseteq X$, $y \in f[\overline{E}]$ and O be neighbourhood of y. We have some $x \in \overline{E}$ such that $f(x) = y$. By continuity of f, $f^{-1}[O]$ is a neighbourhood of x. Then $E \cap f^{-1}[O] \neq \emptyset$, so $f[E \cap f^{-1}[O]] \neq \emptyset$. Hence $f[E] \cap O \neq \emptyset$ (because $f[E \cap f^{-1}[O]] \subseteq f[E] \cap O$). Then, $y \in \overline{f[E]}$. Therefore $f[\overline{E}] \subseteq \overline{f[E]}$. Thus, $\overline{f[\overline{E}]} = \overline{f[E]}$.

This equality ensures the composition by the fact $Z \setminus \overline{g[f[X \setminus U]]} = Z \setminus \overline{g[\overline{Y \setminus [Y \setminus \overline{f[X \setminus U]}]}]}$.

2.3.3 Remark. We often define the morphism mapping to respect the domains and codomains by construction. So, sometimes we don't need to explicitly check if it respects domains and codomains.

2.3.4 Example. Let $\mathcal{A}, \mathcal{B}$ be two categories and X an object of $\mathcal{B}$, we can define a **constant functor** $\Delta_X : \mathcal{A} \longrightarrow \mathcal{B}$ maps $A \mapsto X$, and any arrow to Id_X; and also the **identity functor** $Id_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ which assigns each object to itself and each morphism to itself. On the other if $\mathcal{A}$ is a subcategory of $\mathcal{B}$, we have an **inclusion functor** $\mathbb{I} : \mathcal{A} \to \mathcal{B}$ which assigns each object to itself and each morphism to itself.

2.3.5 Definition. A **forgetful functor** is a functor that is defined by 'forgetting' some structure. For example, the forgetful functor from $\mathbf{Grp}$ to $\mathbf{Set}$ forgets the group structure of a group, remembering only the underlying set and map.

2.3.6 Definition. A **contravariant functor** $: \mathcal{D} \longrightarrow \mathcal{C}$ consists a functor $\mathbb{F} : D^{op} \longrightarrow \mathcal{C}$.

2.3.7 Example. $\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm}$

$$\begin{aligned} \Omega: \quad & \mathbf{Top} \to \mathbf{Frm} \\ & (X,\tau_X) \mapsto \Omega(X,\tau_X) = (\tau_X, \subseteq) \\ (X,\tau_X) \xrightarrow{f} (Y,\tau_Y) & \mapsto (\tau_Y,\subseteq) \xrightarrow{\Omega(f)} (\tau_X,\subseteq) \text{ with } \Omega(f) : U \mapsto f^{-1}[U]. \end{aligned} \tag{2.3.2}$$

For each object, it is well-defined by Example 2.2.8. We know that $f^{-1}[-]$ preserves inclusion, finite intersection, and arbitrary union then it is a $\mathbf{Frm}$-morphism. Moreover, Ω respects the composition because

$$\forall W \in \Omega(Z), (g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]] \quad (if\ \ (Y,\tau_Y) \xrightarrow{g} (Z,\Omega(Z))\ \ in\ \ \mathbf{Top}).$$

2.3.8 Definition (Picado and Pultr (2011))**.** Let $\mathcal{K}$ be a small category and $\mathcal{C}$ a general one. A ($\mathcal{K}$-)**diagram** (in $\mathcal{C}$) is a functor $\mathbb{D} : \mathcal{K} \longrightarrow \mathcal{C}$. When we say diagram, it means that the domain is small.

2.3.9 Definition. A functor $\mathbb{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is

(a) faithful if for each $A, B \in \mathcal{D}$, the map $\mathbb{F} : \mathcal{D}(A, B) \longrightarrow \mathcal{C}(\mathbb{F}(A), \mathbb{F}(B))$ is injective.

(b) full if for each $A, B \in \mathcal{D}$, the map $\mathbb{F} : \mathcal{D}(A, B) \longrightarrow \mathcal{C}(\mathbb{F}(A), \mathbb{F}(B))$ is surjective.

(c) embedding if it is faithful and the map: $\mathbb{F} : Ob(\mathcal{D}) \longrightarrow Ob(\mathcal{C})$ is injective. Such $\mathbb{F}$ identifies the $\mathcal{D}$ as a subcategory of the $\mathcal{C}$.

2.3.10 Definition. The subcategory $\mathcal{D}$ of $\mathcal{C}$ is said full subcategory if the inclusion functor is full.

2.3.11 Example. The $\mathbf{Grp}$ is a full subcategory of $\mathbf{Mon}$ but $\mathbf{Mon}$ is not a full subcategory of $\mathbf{Sgrp}$ because any $\mathbf{Sgrp}$-morphism between two monoids does not always preserve the neutral element.

2.3.12 Remark. Any functor preserves isomorphism and if it is full and faithful, it reflects isomorphism which means the image of f is isomorphism implies f is isomorphism for any f in the domain category.

2.3.13 Definition. Let $\mathbb{F}, \mathbb{G} : \mathcal{D} \longrightarrow \mathcal{C}$ two functors. A **natural transformation**

$\alpha : \mathbb{F} \overset{\bullet}{\Longrightarrow} \mathbb{G}$ is a collection of morphisms in $\mathcal{C}$ $\alpha = (\alpha_A : \mathbb{F}(A) \longrightarrow \mathbb{G}(A))_{A \in \mathcal{D}}$ such that for any

$$\begin{array}{c} A \\ {\scriptstyle f}\big\downarrow {\scriptstyle \text{ in } \mathcal{D}} \\ B \end{array} \quad \text{the square:} \quad \begin{array}{ccc} \mathbb{F}(A) & \xrightarrow{\alpha_A} & \mathbb{G}(A) \\ {\scriptstyle \mathbb{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{G}(f)} \\ \mathbb{F}(B) & \xrightarrow{\alpha_B} & \mathbb{G}(B) \end{array} \quad \text{commutes.}$$

We say α is a **natural ismorphism** if each α_A is an isomorphism. In this case, we write $\mathbb{F} \cong \mathbb{G}$.

2.3.14 Example. We can verify that $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}, S \mapsto \mathcal{P}(S), f \mapsto f[-]$ is a functor. If we define $\eta_S : S \to \mathcal{P}(S), x \mapsto \{x\}$ for each set S, we can get $\eta := (\eta_S)_{S \in \mathbf{Set}}$ is a natural transformation

$$\eta : Id_{\mathbf{Set}} \overset{\bullet}{\Longrightarrow} \mathcal{P}.$$

2.3.15 Definition. The category $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** if there exist two functors $\mathcal{A} \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathcal{B}$ and two natural ismorphisms $\alpha : Id_{\mathcal{A}} \overset{\bullet}{\Longrightarrow} \mathbb{GF}$ and $\beta : \mathbb{FG} \overset{\bullet}{\Longrightarrow} Id_{\mathcal{B}}$. On other, they are **isomorphic** if there exist two functors $\mathcal{A} \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathcal{B}$ such that $Id_{\mathcal{A}} = \mathbb{GF}$ and $Id_{\mathcal{B}} = \mathbb{FG}$. In this case, we write $\mathcal{A} \cong \mathcal{B}$.

From these definitions, isomorphic implies equivalent but the converse is false in general. For instance, let's take $\mathcal{A}$ be the discrete category with one object a and $\mathcal{B}$ is a category with only two objects b, c and only two no-identity morphisms $b \underset{g}{\overset{f}{\rightleftarrows}} c$. The functor $\mathbb{F} : \mathcal{A} \to \mathcal{B}, a \mapsto b$ and the only functor $\mathbb{G}$ from $\mathcal{B}$ to $\mathcal{A}$ give us the equivalence of $\mathcal{A}$ and $\mathcal{B}$ but we know clearly that they are not isomorphic.

2.3.16 Definition. Given two categories $\mathcal{D}$ and $\mathcal{C}$, $[\mathcal{D}, \mathcal{C}]$ (or else $\mathcal{C}^{\mathcal{D}}$) is a category whose objects are functors $\mathcal{D} \to \mathcal{C}$ and whose arrows are natural transformations between these functors.

It is easy to check that it is a category.

2.4 Universal properties, representability, and the Yoneda lemma

2.4.1 Definition (initial, final object)**.** Let $\mathcal{C}$ be a category and A be an object.

(a) A is an **initial object** if for every B object of $\mathcal{C}$ there exists an unique morphism f from A to B. That means $\mathcal{C}(A, B)$ is a singleton for any $B \in \mathcal{C}$.

(b) A is a **final object** if for every B object of $\mathcal{C}$ there exists an unique morphism f from B to A. That means $\mathcal{C}(B, A)$ is a singleton for any $B \in \mathcal{C}$.

We can see that the notion of the final object is dual to the notion of the initial object.

2.4.2 Proposition. Let $\mathcal{C}$ be a category and A, B be two objects of $\mathcal{C}$.

(a) If A and B are both initial objects then A and B are ismorphic.
Indeed, we must have a morphism from A to B $f : A \longrightarrow B$ and a morphism from B to A $g : B \longrightarrow A$. Then we have $fg : B \longrightarrow B$ and $gf : A \longrightarrow A$ that must be $\mathbf{1}_B$ and $\mathbf{1}_A$ by uniqueness of the element of $\mathcal{C}(B, B)$ and $\mathcal{C}(A, A)$.

(b) If A and B are both final objects then A and B are ismorphic.
By duality.

2.4.3 Definition. Let $\mathcal{C}$ be a locally small category and $A \in \mathcal{C}$. We define a functor, called hom-set functor $\mathcal{C}^A = \mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ which assigns $B \mapsto \mathcal{C}(A, B)$ and $f \mapsto \mathcal{C}(A, f)$ where $\mathcal{C}(A, f)(g) = fg$. We denote $\mathcal{C}(A, f) = f_*$. We also have dually a functor $\mathcal{C}(-, A)$ denoted by $f^* = \mathcal{C}(f, A) : g \mapsto gf$.

2.4.4 Definition. Let $\mathcal{C}$ be a locally small category. A functor $\mathbb{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ is representable if $\mathbb{F}$ is naturally isomorphic to $\mathcal{C}^A$ for some $A \in \mathcal{C}$.

2.4.5 Example. The forgetful functor $\mathbb{U}$: **Grp** $\longrightarrow$ **Set** is represented by the group $\mathbb{Z}$. For any group G, we can consider the map $\alpha_G : G = \mathbb{U}(G) \longrightarrow \textbf{Grp}(\mathbb{Z}, G)$, $g \mapsto (n \mapsto \alpha_G(g)(n) = g^n)$. It is clear that the correspondence $(n \mapsto g^n)$ is a group homomorphism for any $g \in G$ then α_G is well-defined. Considering the map $\beta_G : \textbf{Grp}(\mathbb{Z}, G) \longrightarrow G$, $f \mapsto f(1)$ which is cleary well-defined. Moreover,

$$\forall g \in G : \beta_G \circ \alpha_G(g) = \beta_G(n \mapsto g^n) = g; \;\; \forall f \in \textbf{Grp}(\mathbb{Z}, G) : \alpha_G \circ \beta_G(f) = \alpha_G(f(1)) = (n \mapsto (f(1))^n = f(n)) = f.$$

which ensure us α_G is a bijection for any G. It remains for us to show that $\alpha = (\alpha_G)_{G \in \textbf{Grp}}$ is a natural transformation. Furthermore, for any $G, H \in$ **Grp**, $f : G \longrightarrow H$, $g \in G$, it is also clear that

$$\alpha_H(f(g)) = (k \mapsto (f(g))^k) = (k \mapsto f(g^k)) = ((m \mapsto f(m)) \circ (k \mapsto g^n)) = f \circ \alpha_G(g) = \textbf{Grp}(\mathbb{Z}, f)(\alpha_G)(g).$$

$$\text{Hence, } \forall G \xrightarrow{f} H \;\; \textit{in} \;\; \mathbf{Grp} : \quad \mathbf{Grp}(\mathbb{Z}, f) \circ \alpha_G = \alpha_H \circ \mathbb{U}(f).$$

2.4.6 Remark. A representable functor preserves monomorphism and a faithful functor reflects monomorphism.

2.4.7 Proposition (Yoneda Lemma)**.** Let $\mathbb{F} : \mathcal{C} \longrightarrow \mathbf{Set}$ be a functor where $\mathcal{C}$ is locally small and $O \in Ob(\mathcal{C})$. Then there is a bijection

$$\rho^O_{\mathbb{F}} : \{\alpha : \mathcal{C}^O \overset{\bullet}{\Longrightarrow} \mathbb{F}\} \longrightarrow \mathbb{F}O, \text{ given by } \rho^O_{\mathbb{F}}(\alpha) := \alpha_O(\mathbf{1}_O).$$

Moreover, this correspondence is natural in both O and $\mathbb{F}$.

Proof. Clearly $\rho^O_{\mathbb{F}}$ is well-defined. For any $x \in \mathbb{F}O$, $\phi x := \mathcal{C}^O \overset{\bullet}{\Longrightarrow} \mathbb{F}$ such that for any $\mathcal{C}$-object A, $\phi x_A : \mathcal{C}(O, A) \to \mathbb{F}A, f \mapsto (\mathbb{F}f)(x)$. For any $\mathcal{C}$-object A, ϕx_A is well-defined.

So $\phi : \mathbb{F}O \to \{\alpha : \mathcal{C}^O \overset{\bullet}{\Longrightarrow} \mathbb{F}\}, x \mapsto \phi x$ is well-defined because ϕ_x is always natural transformation by

$$\mathcal{C}(O, \mathbb{F}m) \circ \phi x_A(n) = \mathbb{F}m[\mathbb{F}n(x)] = [\mathbb{F}(\mathcal{C}(O, m)(n)](x) = \phi x_B \circ \mathcal{C}(O, m)(n)$$

for any $A \xrightarrow{m} B, O \xrightarrow{n} A$ ***in*** $\mathcal{C}$. Moreover, for any $x \in \mathbb{F}O$, $\rho^O_{\mathbb{F}} \circ \phi(x) = \phi x_O(\mathbf{1}_O) = (\mathbb{F}\mathbf{1}_O)(x) = x$.

In addition, for any $\alpha : \mathcal{C}^O \overset{\bullet}{\Longrightarrow} \mathbb{F}$:

$$\forall A \in \mathcal{C}, O \xrightarrow{m} A : \quad (\phi \circ \rho^O_{\mathbb{F}}(\alpha))_A(m) = \Big(\phi(\alpha_O(\mathbf{1}_O))\Big)_A(m) = \mathbb{F}m(\alpha_O(\mathbf{1}_O)) = \alpha_A(m).$$

Hence, the bijection. It remains for us to prove the naturality in O and in $\mathbb{F}$. For $O \xrightarrow{m} A$ ***in*** $\mathcal{C}$

$$\forall \alpha : \mathcal{C}^O \overset{\bullet}{\Longrightarrow} \mathbb{F} : \mathbb{F}m \circ \rho^O_{\mathbb{F}}(\alpha) = \mathbb{F}m(\alpha_O(\mathbf{1}_O)) = \alpha_A(m) = \rho^A_{\mathbb{F}}\Big((f \mapsto \alpha_B(fm))_{B \in \mathcal{C}}\Big) = \rho^A_{\mathbb{F}} \circ (m^*)^{*1}(\alpha).$$

Then we have naturality in O. On the other hand, for any $\beta : \mathbb{F} \overset{\bullet}{\Longrightarrow} \mathbb{G}$, $\alpha : \mathcal{C}^O \overset{\bullet}{\Longrightarrow} \mathbb{F}$,

$$\beta_O \circ \rho^O_{\mathbb{F}}(\alpha) = \beta_O \alpha_O(\mathbf{1}_O) = (\beta\alpha)_O(\mathbf{1}_O) = \rho^O_{\mathbb{G}} \circ \beta_*(\alpha).$$

Thus the naturality in $\mathbb{F}$. □

This lemma is useful, has many interpretations such in representability and the fact that any locally small category $\mathcal{C}$, there is a faithful and full embedding from $\mathcal{C}$ to $\mathbf{Set}^{\mathcal{C}^{op}}$.

2.5 Some constructions

2.5.1 Products, coproducts

2.5.2 Definition. Let $\mathcal{S}$ be a family of objects of a category $\mathcal{C}$. Let us construct a category $\mathcal{C}(\mathcal{S})$ whose objects are pairs (O, m), where O is an object of $\mathcal{C}$ and m is a collection of morphisms of domain O and codomain in $\mathcal{S}$ in the sense: $m = (m_Q : O \longrightarrow Q)_{Q \in \mathcal{S}}$.
The morphisms from (A, m) to (B, n) are all morphisms $f : A \longrightarrow B$ such that:

$$\forall Q \in \mathcal{S} : \quad n_Q \circ f = m_Q.$$

We can easily check that it is a category (composition is like in $\mathcal{C}$).

2.5.3 Definition. A **product** of $\mathcal{S}$ is a final object (P, p) of a category $\mathcal{C}(\mathcal{S})$, often denoted

$$\prod_{Q \in \mathcal{S}} Q, \qquad \text{the morphism } (p_Q)_{Q \in \mathcal{S}} \text{ are the projections.}$$

In particular if $\mathcal{S} = \{X, Y\}$, the product of $\mathcal{S}$ is often written $X \times Y$(binary product).
Dually, a coproduct of $\mathcal{S}$ $\coprod_{Q \in \mathcal{S}} Q$ is collection of morphisms $g = (g_Q : Q \to C)_{Q \in \mathcal{S}}$ such that for any collection $m = (m_Q : Q \to A)_{Q \in \mathcal{S}}$, there is an unique $f : C \to A$ such that $fg_Q = m_Q$ for all $s \in \mathcal{S}$.

[1] m^* is about hom-set functor defined at Definition 2.4.3 but the second star in $(m^*)^*$ is something like abuse extension of this concept for an natural transformation. It assigns $\alpha = (\alpha_A)_{A \text{ object}}$ to $\alpha m^* := (f \mapsto \alpha_A(m^*(f)))_{A \text{ object}}$.

2.5.4 Remark. Products do not always exist. The discrete category can be taken as a counter-example. But, by Proposition 2.4.2, if the product exists it is unique up to isomorphism.

2.5.5 Example. We consider the poset $(\mathbb{R}, \leq)$ as a category and S a set of some object of $\mathbb{R}$ ($S \subset \mathbb{R}$) which is bounded below by α. For any $a \in \mathbb{R}$, $(a, m) \in \mathcal{C}(S)$ for some m, means clearly that a is a lower bound of S. By definition of infinima of S, α is lower bound of S: $\alpha \leq s$ for each $s \in S$ and for any lower bound α' of S: $\alpha' \leq \alpha$. That ensure us $(\alpha, m) \in \mathcal{C}(S)$ for some family m of morphisms and for all $(a, m') \in \mathcal{C}(S)$ there is a morphism (a, m') to (α, m) which is clearly unique by the definition of category $(\mathbb{R}, \leq)$. Hence (α, m) is the product of S.
It can be generalized in all posets as a category.

2.5.6 Equalizers, coequalizers

Let's introduce a terminology in Leinster (2014): a fork in $\mathcal{C}$ on t and s is the objects and morphisms

$$O \xrightarrow{m} X \underset{t}{\overset{s}{\rightrightarrows}} Y \quad \text{such that } tm = sm.$$

2.5.7 Definition. Let $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ be two paralel morphisms in a category $\mathcal{C}$. Let's construct a category of forks on t and s, $\mathcal{CF}[s, t] = \mathcal{CF}[t, s]$ which the objects are all forks of type $O \xrightarrow{m} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ and the morphisms from $A \xrightarrow{m} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ to $B \xrightarrow{n} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ are all morphisms $f : A \longrightarrow B$ in $\mathcal{C}$ such that

$$m = n \circ f.$$

We can easily check that it is a category (composition is like in $\mathcal{C}$).

2.5.8 Definition. For $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ two parallel morphisms in a category $\mathcal{C}$, an **equalizer** of t and s is a morphism $E \xrightarrow{e} X$ in $\mathcal{C}$ such that $E \xrightarrow{e} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is a final object for the category $\mathcal{CF}[s, t]$, we denote it $Equ(s, t)$.
Dually, we define coequalizers but I want to define it differently in Example 2.6.10.

2.5.9 Example. In **Set**, for two parallel maps $X \underset{t}{\overset{s}{\rightrightarrows}} Y$, we can consider the set
$E = \{x \in X / s(x) = t(x)\}$ with the inclusion map $i : E \hookrightarrow X$. By construction $E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is a fork on s and t. It is universal among all forks on s and t (final object). Indeed for each $A \xrightarrow{f} X$, if $sf = tf$ (we have $f[A] \subseteq E$), then $\bar{f} : A \to E$, $a \mapsto \bar{f}(a) = f(a)$ is well defined and $i\bar{f} = f$ which means that $\bar{f}$ is a morphism of forks from to $A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ to $E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$.
If $(A \xrightarrow{f} X) \xrightarrow{g} (E \xrightarrow{i} X)$ in $\mathcal{CF}[t, s]$, then $ig = f$ and for all $a \in A : ig(a) = f(a)$. Hence $g = \bar{f}$.
Hence $E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is a final object of the category $\mathcal{CF}[s, t]$ then equalizer of $X \underset{t}{\overset{s}{\rightrightarrows}} Y$.

2.5.10 Example. Let $G \xrightarrow{f} H$ be in **Grp**, $\theta : G \longrightarrow H, x \mapsto e_H$ and , we have $Ker(f)$ be the kernel of f. Then $Equ(f, \theta) = Ker(f) \xrightarrow{i} G \underset{\theta}{\overset{f}{\rightrightarrows}} H$ where i is the inclusion.

2.5.11 Pullbacks, pushouts

2.5.12 Definition. Let $A \xrightarrow{s} O, B \xrightarrow{t} O$ two morphisms in a category $\mathcal{C}$. Let's construct a category $\mathcal{C}\mathbb{P}[t, s]$ which the objects are all pairs of morphisms $(X \xrightarrow{m} A, X \xrightarrow{n} B)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "n"'] & B \arrow[d, "s"] \\ A \arrow[r, "t"'] & O \end{tikzcd}$$

commutes and the morphisms from $(X \xrightarrow{m} A, X \xrightarrow{n} B)$ to $(Y \xrightarrow{m'} A, X \xrightarrow{n'} B)$ are all morphisms $f : X \longrightarrow Y$ in $\mathcal{C}$ such that $m = m'f$ and $n = n'f$.
We can easily check that it is a category (composition is like in $\mathcal{C}$).

2.5.13 Definition. A **pullback** of $A \xrightarrow{s} O, B \xrightarrow{t} O$ is a final object of the category $\mathcal{C}\mathbb{P}[t, s]$.
Dually, a pushout of $A \overset{s}{\leftarrow} O \overset{t}{\rightarrow} B$ is $A \overset{\bar{s}}{\rightarrow} P \overset{\bar{t}}{\leftarrow} B$ such that $\bar{t}t = \bar{s}s$ and for every $A \overset{f}{\rightarrow} X \overset{g}{\leftarrow} B$ satisfying $gt = fs$ there is an unique $m : P \rightarrow X$ in $\mathcal{C}$ such that $m\bar{t} = g$ and $m\bar{s} = f$.

2.5.14 Example. In **Set**, for $A \xrightarrow{s} O, B \xrightarrow{t} O$, $E := \{(x, y) \in A \times B : s(x) = t(y)\}$. We can verify that $(E \xrightarrow{q_A} A, E \xrightarrow{q_B} B)$ is a pullback of $A \xrightarrow{s} O, B \xrightarrow{t} O$ (q_A, q_B are the projections' restrictions).

2.5.15 Proposition (Awodey (2010) Corollary 5.6)**.** If a category $\mathcal{C}$ has binary products and equalizers, then it has pullbacks.

Proof. Let $A \xrightarrow{s} O, B \xrightarrow{t} O$ ***in*** $\mathcal{C}$. We get $P = A \times B \underset{to p_B}{\overset{so p_A}{\rightrightarrows}} O$ and the equalizer of this parallel pair be (E, e). We can verify that $(E \xrightarrow{p_A \circ e} A, E \xrightarrow{p_B \circ e} B)$ is a final object of $\mathcal{C}\mathbb{P}[t, s]$. $\square$

2.6 Limits and colimits

To establish limits (with colimits), let's define the lower bound (with upper bound) according to the terms used by Picado and Pultr (2011).

2.6.1 Definition. Consider a $\mathcal{K}$-diagram $\mathbb{D} : \mathcal{K} \longrightarrow \mathcal{C}$. A **lower bound** of $\mathbb{D}$ is an object X of $\mathcal{C}$ together with a collection of morphisms in $\mathcal{C}$ $(f_a : X \longrightarrow \mathbb{D}(a))_{a \in \mathcal{K}}$ such that :

$$\begin{tikzcd} a \arrow[d, "\forall\alpha \text{ in } \mathcal{K}"'] & \mathbb{D}a \arrow[d, "\mathbb{D}(\alpha)"'] & \\ b & \mathbb{D}b & X \arrow[ul, "f_a"'] \arrow[l, "f_b"] \end{tikzcd}$$

commutes. X is said vertex.

2.6.2 Remark (Category of lower bounds of $\mathbb{D}$)**.** A morphism of lower bound (of $\mathbb{D}$) from $(f_a : X \longrightarrow \mathbb{D}(a))_{a\in\mathcal{K}}$ to $(g_a : Y \longrightarrow \mathbb{D}(a))_{a\in\mathcal{K}}$ is a morphism $m : X \longrightarrow Y$ in $\mathcal{C}$ such that

$$\forall a \in \mathcal{K} : \quad f_a = g_a \circ m.$$

We can easily check that it is a category (composition is like in $\mathcal{C}$).

2.6.3 Remark (Riehl (2017) p74)**.** Considering the constant functor $\Delta_X : \mathcal{K} \longrightarrow \mathcal{C}$ (see Example 2.3.4), a cone (or lower bound) of $\mathbb{D} : \mathcal{K} \longrightarrow \mathcal{C}$ is a natural transformation $(X, \alpha_X : \Delta_X \overset{\bullet}{\Longrightarrow} \mathbb{D})$.

2.6.4 Example. Let's consider the poset $(P, \leq)$ as a category. Be A a subset of P. Since $(A, \leq)$ is naturally a poset with the induced order. Consider the inclusion maps $\mathbb{I}$ as a functor (easy to verify). A lower bound of a functor $\mathbb{I}$ is none other than the usual lower bound of A in the poset.

2.6.5 Definition. A limit of $\mathbb{D}$ is a final object of the category of lower bounds of $\mathbb{D}$.

2.6.6 Example. If we return to Example 2.6.4, we can see that if P is a (complete) lattice, the limit of the inclusion functor $\mathbb{I}_A : A \longrightarrow P$ is the $\bigwedge A$.

The upper bound of $\mathbb{D}$ and colimit are defined dually.

2.6.7 Definition. Let a $\mathcal{K}$-diagram $\mathbb{D} : \mathcal{K} \longrightarrow \mathcal{C}$. A **upper bound** of $\mathbb{D}$ is an object X of $\mathcal{C}$ together with a collection of morphisms in $\mathcal{C}$ $(f_a : \mathbb{D}(a) \longrightarrow X)_{a\in\mathcal{K}}$ such that :

$$\forall a \xrightarrow{\alpha} b \ \textbf{\textit{in}} \ \mathcal{K} : \quad f_a = f_b \circ \mathbb{D}\alpha.$$

2.6.8 Remark (Category of upper bounds of $\mathbb{D}$)**.** A morphism of lower bounds (of $\mathbb{D}$) from $(f_a : \mathbb{D}(a) \longrightarrow X)_{a\in\mathcal{K}}$ to $(g_a : \mathbb{D}(a) \longrightarrow Y)_{a\in\mathcal{K}}$ is a morphism $m : X \longrightarrow Y$ in $\mathcal{C}$ such that for all a object of $\mathcal{K}$, $g_a = m \circ f_a$. We can easily check that it is a category (composition is like in $\mathcal{C}$).

2.6.9 Definition. A colimit of $\mathbb{D}$ is an initial object of the category of upper bounds of $\mathbb{D}$.

2.6.10 Example. Let $\mathcal{F}$ be the category in which the objects are x and y and only non-identity arrows are $x \underset{t'}{\overset{s'}{\rightrightarrows}} y$. Now, let's consider a $\mathcal{F}$-diagram $\mathbb{D} : \mathcal{F} \longrightarrow \mathcal{C}$ which sends $x \mapsto X$, $y \mapsto Y$, $s' \mapsto s$ and $t' \mapsto t$. The upper bound of $\mathbb{D}$ are $(X \overset{f}{\to} V \overset{g}{\leftarrow} Y)$ such that $f = gs$ and $f = gt$, thus, $gs = gt$. So, we can forget f and think only that a upper bound of $\mathbb{D}$ are $Y \xrightarrow{g} V$ in $\mathcal{C}$ satisfying $gs = gt$.
Then, colimit of $\mathbb{D}$ is a morphism $(Y \overset{a}{\to} O)$ in $\mathcal{C}$ such that $as = at$ and for every $(Y \overset{g}{\to} V)$ verifying $gs = gt$, there is an unique $O \xrightarrow{m} V$ such that $ma = g$.
This colimit is none other than the coequalizer of $X \underset{t}{\overset{s}{\rightrightarrows}} Y$, which is dual of Definition 2.5.8.

2.6.11 Remark. The limit and colimit of a diagram do not always exist. But if they exist, they are unique up to isomorphism by Proposition 2.4.2.

We could verify that all of the constructions at 2.5 are particular cases of limits and colimits.

2.6.12 Definition. A category $\mathcal{C}$ is complete (resp. cocomplete) if each diagram in $\mathcal{C}$ has a limit (resp. colimit).

2.6.13 Theorem (Freyd-Maranda Picado and Pultr (2011)). A category $\mathcal{C}$ is complete if and only if it has all products and equalizers.

This proof is not similar to the proof in Picado and Pultr (2011) but I will detail the one in the Lectures by Peter Johnstone, Category Theory at Cambridge University, Part III Michaelmas 2015.

Proof. For the direction left to right, for any $\mathcal{S}$ family of objects in $\mathcal{C}$, we can consider the category $\mathcal{I}_{\mathcal{S}}$ whose $Ob(\mathcal{I}_{\mathcal{S}}) = \mathcal{S}$ and there is no non-identity arrow and for any parallel arrows $X \overset{s}{\underset{t}{\rightrightarrows}} Y$, we can consider the category $\mathcal{E}$ whose $Ob(\mathcal{E}) = \{X, Y\}$ and the only non-identity arrows are $X \overset{s}{\underset{t}{\rightrightarrows}} Y$. Now, define the diagram (functor) $\mathbb{D} : \mathcal{I}_{\mathcal{S}} \to \mathcal{C}$ wich assigns $S \mapsto S, Id_S \mapsto Id_S$ for any $S \in \mathcal{S}$ and $\mathbb{E} : \mathcal{E} \to \mathcal{C}$ wich assigns $X \mapsto X, Y \mapsto Y, s \mapsto s$ and $t \mapsto t$. We can see that the product of S and the equalizer of $X \overset{s}{\underset{t}{\rightrightarrows}} Y$ are the limits of $\mathbb{D}$ and $\mathbb{E}$ which exist.
For the direction right to left, for any diagram $\mathbb{D} : \mathcal{K} \longrightarrow \mathcal{C}$, then we consider two products:

$$\prod_{i\in\mathcal{K}} \mathbb{D}i = (P \xrightarrow{\pi_i} \mathbb{D}i)_{i\in\mathcal{K}} \text{ ie } \forall (A \xrightarrow{a_i} \mathbb{D}i)_{i\in\mathcal{K}}, \exists! A \xrightarrow{h} P : (\forall i \in \mathcal{K} : \pi_i \circ h = a_i); \tag{2.6.1}$$

$$\prod_{\alpha \text{ in } \mathcal{K}} \mathbb{D}cod(\alpha) = (Q \xrightarrow{\rho_\alpha} \mathbb{D}cod(\alpha)) \text{ ie } \forall (B \xrightarrow{b_\alpha} \mathbb{D}cod(\alpha))_{\alpha \text{ in } \mathcal{K}}, \exists! B \xrightarrow{n} P(\forall \alpha : \rho_\alpha \circ n = b_\alpha). \tag{2.6.2}$$

If we define $(f_\alpha := \pi_{cod(\alpha)} : P \to \mathbb{D}cod(\alpha))_{\alpha \textit{ in } \mathcal{K}}$ and $(g_\alpha := \mathbb{D}\alpha \circ \pi_{dom(\alpha)} : P \to \mathbb{D}cod(\alpha))_{\alpha \textit{ in } \mathcal{K}}$, (2.6.2) tells us that there exist an unique $f : P \to Q$ and an unique $g : P \to Q$ such that

$$\forall \alpha \textit{ in } \mathcal{K} : \quad \rho_\alpha f = f_\alpha \quad \text{and} \quad \rho_\alpha g = g_\alpha. \tag{2.6.3}$$

Now, let (L, λ) be the equalizer of $P \overset{f}{\underset{g}{\rightrightarrows}} Q$ and denote $\lambda_i = \pi_i \circ \lambda : L \to \mathbb{D}i$ for each $i \in \mathcal{K}$.
For any $i \xrightarrow{\alpha} j$ in $\mathcal{K}$, since (L, λ) is the equalizer: $g_\alpha \lambda = f_\alpha \lambda$, so $(\mathbb{D}\alpha)\lambda_i = (\mathbb{D}\alpha)\pi_i\lambda = \pi_j\lambda = \lambda_j$. Then $(\lambda_i : L \to \mathbb{D}i)_{i\in\mathcal{K}}$ is a lower bound of $\mathbb{D}$. Next, let's take an arbitrary lower bound $(\mu_i : X \to \mathbb{D}i)_{i\in\mathcal{K}}$. Since (2.6.1), there is an unique $X \xrightarrow{\mu} P$ such that for all $i \in \mathcal{K}$, $\pi_i\mu = \mu_i$ $(\star)$.
Let $i \xrightarrow{\alpha} j$ *in* $\mathcal{K}$. Since $(\mu_i)_{i\in\mathcal{K}}$ is lower bound, $(\mathbb{D}\alpha)\mu_i = \mu_j$ $(\star\star)$. Then

$$g_\alpha\mu = (\mathbb{D}\alpha)\pi_i\mu \overset{(\star)}{=} (\mathbb{D}\alpha)\mu_i \overset{(\star\star)}{=} \mu_j \overset{(\star)}{=} \pi_j\mu = f_\alpha\mu. \tag{2.6.4}$$

Considering $(f_\alpha\mu : X \to \mathbb{D}cod(\alpha))_{\alpha \text{ in } \mathcal{K}}$ and $(g_\alpha\mu : X \to \mathbb{D}cod(\alpha))_{\alpha \text{ in } \mathcal{K}}$ with (2.6.2) there are an unique $X \xrightarrow{n} Q$ and unique $X \xrightarrow{n'} Q$ such that $\rho_\alpha \circ n = f_\alpha\mu$ and $\rho_\alpha \circ n' = g_\alpha\mu$ for any $i \xrightarrow{\alpha} j$ *in* $\mathcal{K}$. By (2.6.3): $f\mu, g\mu$ satisfy such n and n'. Combining this fact with (2.6.4), $\rho_\alpha f\mu = \rho_\alpha g\mu$. The uniqueness ensures the equality $g\mu = f\mu$. Hence, $X \xrightarrow{\mu} P \overset{f}{\underset{g}{\rightrightarrows}} Q$ is a fork in $\mathcal{C}$.

Since (L, λ) is the equalizer of $P \overset{f}{\underset{g}{\rightrightarrows}} Q$, then there is an unique $X \xrightarrow{m} L$ in $\mathcal{K}$ such that $\mu = \lambda m$. Then for any $i \in \mathcal{K} : \mu_i = \pi_i\mu = \pi_i\lambda m = \lambda_i m$. (2.6.1) ensures us also that $\pi_i\mu = \pi_i\lambda m'$ for each $i \in \mathcal{K}$ implies $\mu = \lambda m'$. So we can say there is unique $X \xrightarrow{m} L$ in $\mathcal{K}$ such that $\mu_i = \lambda_i m$ for each $i \in \mathcal{K}$. Therefore $(L, \lambda_i)_{i\in\mathcal{K}}$ is a (the) limit of $\mathbb{D}$. $\square$

2.7 Adjunction

2.7.1 Definition. An adjunction consists of a pair of functors $\mathcal{A} \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathcal{B}$ with the isomorphism ε

$$\Big(\varepsilon_{A,B} : \mathcal{B}(\mathbb{F}A, B) \cong \mathcal{A}(A, \mathbb{G}B)\Big)_{A\in\mathcal{A}, B\in\mathcal{B}} \quad \text{that is natural in both } A \text{ and } B. \tag{2.7.1}$$

Naturality means the commutativity of the diagrams, for each $A' \xrightarrow{m} A$ ***in*** $\mathcal{A}$ and $B \xrightarrow{m'} B'$ ***in*** $\mathcal{B}$,

$$\begin{array}{ccc} \mathcal{B}(\mathbb{F}A, B) & \xrightarrow{\varepsilon_{A,B}} & \mathcal{A}(A, \mathbb{G}B) \\ {\scriptstyle (\mathbb{F}m)^*}\downarrow & & \downarrow{\scriptstyle m^*} \\ \mathcal{B}(\mathbb{F}A', B) & \xrightarrow[\varepsilon_{A',B}]{} & \mathcal{A}(A', \mathbb{G}B); \end{array} \qquad \begin{array}{ccc} \mathcal{B}(\mathbb{F}A, B) & \xrightarrow{\varepsilon_{A,B}} & \mathcal{A}(A, \mathbb{G}B) \\ {\scriptstyle m'_*}\downarrow & & \downarrow{\scriptstyle (\mathbb{G}m')_*} \\ \mathcal{B}(\mathbb{F}A, B') & \xrightarrow[\varepsilon_{A,B'}]{} & \mathcal{A}(A, \mathbb{G}B'). \end{array}$$

Naturality in A and naturality in B can be expressed explicitly:

$$\forall A' \xrightarrow{m} A \text{ in } \mathcal{A}, f \in \mathcal{B}(\mathbb{F}A, B) : \varepsilon_{A,B}(f) \circ m = \varepsilon_{A',B}(f \circ \mathbb{F}m), \text{ and} \tag{2.7.2}$$

$$\forall B \xrightarrow{m'} B' \text{ in } \mathcal{B}, f' \in \mathcal{B}(\mathbb{F}A, B) : \mathbb{G}m' \circ \varepsilon_{A,B}(f') = \varepsilon_{A,B'}(m' \circ f'). \tag{2.7.3}$$

In this case, we say that $\mathbb{F}$ is a left adjoint to $\mathbb{G}$ or $\mathbb{G}$ is a right adjoint to $\mathbb{F}$ and we write $\mathbb{F} \dashv \mathbb{G}$.

2.7.2 Example. $\mathbf{Ab}$ be the category of the Abelian groups. The functor abelianization is $\mathbf{Grp} \xrightarrow{\mathbb{A}} \mathbf{Ab}$

$$\begin{aligned} \mathbb{A} : \quad & \mathbf{Grp} \to \mathbf{Ab} \\ & G \mapsto \mathbb{A}(G) = G/[G,G] \\ & G \xrightarrow{f} H \mapsto G/[G,G] \overset{\mathbb{A}(f)=\mathbb{A}_f}{\longrightarrow} H/[H,H] \text{ with } \mathbb{A}_f(\bar{g}) = \overline{f(g)} \text{ where } \bar{X} \text{ be the class of } X. \end{aligned} \tag{2.7.4}$$

Firstly $G/[G,G]$ is known as an Abelian group. So the object mapping is well-defined.

$\mathbb{A}_f$ is well-defined because if $\bar{g_1} = \bar{g_2}$, $g_1g_2^{-1} \in [G,G]$ i.e $g_1g_2^{-1} = \prod a_ib_ia_i^{-1}b_i^{-1}$ which implies $f(g_1)f(g_2)^{-1} = \prod f(a_i)f(b_i)f(a_i)^{-1}f(b_i)^{-1} \in [H,H]$ and $\overline{f(g_1)} = \overline{f(g_2)}$. Moreover, $\mathbb{A}_f$ is group homomorphism then an $\mathbf{Ab}$-morphism. So the arrow mapping is well-defined. Clearly, in addition, it respects identity, domain, codomains and composition. Thus the functoriality.

Recall that is always the canonical surjection $\pi_G : G \to G/[G,G], g \mapsto \bar{g}$.

Let $\mathbb{I} : \mathbf{Ab} \to \mathbf{Grp}$ be the inclusion functor. Now let's fix two objects $A \in \mathbf{Ab}$ and $G \in \mathbf{Grp}$ with $f : G \longrightarrow \mathbb{I}A$ in $\mathbf{Grp}$. It is clear that $\phi_f : \mathbb{A}G = G/[G,G] \longrightarrow A, \bar{g} \mapsto f(g)$ is well-defined (because $\overline{g_1} = \overline{g_2}$ implies $f(g_1)\big(f(g_2)\big)^{-1} = e_A$ by commutativity of A) and verifies $\phi_f \circ \pi_G = f$. In addition, for any $h \in \mathbf{Ab}(\mathbb{A}G, A)$, $f_h = h \circ \pi_G$ is in $\mathbf{Grp}(G, \mathbb{I}A)$ and verify $\phi_{f_h} = h$, hence the surjection of ϕ_-. Moreover, for any $f_1, f_2 \in \mathbf{Grp}(G, \mathbb{I}A)$ if $f_1 \neq f_2$, $f_1(g) \neq f_2(g)$ for some $g \in G$. Then $\phi_{f_1}(\overline{g}) \neq \phi_{f_2}(\overline{g})$ for some $\overline{g} \in G/[G,G] = \mathbb{A}G$, so, $\phi_{f_1} \neq \phi_{f_2}$. Hence the injection of ϕ_-.

Thus, for each $G \in \mathbf{Grp}$ and $A \in \mathbf{Ab}$ we have a bijection $\varepsilon_{G,A} = \phi_{A,G}^{-1} : \mathbf{Ab}(\mathbb{A}G, A) \cong \mathbf{Grp}(G, \mathbb{I}A)$ which maps h to $h \circ \pi_G$. To obtain the adjunction, it remains the naturality.

Let $H \xrightarrow{m} G$ **in** $\mathbf{Grp}$ and $\mathbb{A}G \xrightarrow{q} A$ **in** $\mathbf{Ab}$.
For any $x \in H$, $q \circ \pi_G \circ m(x) = q\big(\overline{m(x)}\big) = q \circ \mathbb{A}_m(\overline{x}) = q \circ \mathbb{A}_m \circ \pi_H(x)$. Then $q \circ \pi_G \circ m = q \circ \mathbb{A}m \circ \pi_H$.

So $\varepsilon_{G,A}(q) \circ m = \varepsilon_{H,A}(q \circ \mathbb{A}m)$. That tells us the naturality in G.
Moreover, for any $A \xrightarrow{m} B$ **in Grp**, $\mathbb{A}G \xrightarrow{q} A$ **in Ab**, $\mathbb{I}m \circ \varepsilon_{G,A}(q) = m \circ q \circ \pi_G = \varepsilon_{G,B}(m \circ q)$. That ensures us the naturality in A. Hence the functor $\mathbb{A}$ is the left adjoint to the inclusion $\mathbb{I}$.

2.7.3 Definition (Riehl (2017) Definition 4.5.12)**.** A reflective subcategory of a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ so that the inclusion admits a left adjoint, called the reflector or localization.

2.7.4 Example. Clearly the inclusion $\mathbb{I} : \mathbf{Ab} \to \mathbf{Grp}$ is full then $\mathbf{Ab}$ is a reflective subcategory of $\mathbf{Grp}$ where $\mathbb{A}$ is the reflector since the adjunction was seen in Example 2.7.2.

2.7.5 Example. We shall show that the Galois connection between poset is a particular case of adjunction. Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ two $\mathbf{Pos}$-morphisms with f is a left adjoint to g. Recall from (2.2.1) that it means, in the sense of Galois connection, for any $a \in A$ and $b \in B$: $f(a) \leq b$ is equivalent to $a \leq g(b)$.

Now, consider the poset $(A, \leq)$ and $(B, \leq)$ as two categories $\mathcal{A}, \mathcal{B}$ as seen at Remark 2.2.2. In addition, f and g can be viewed as functors. Indeed, for $a \xrightarrow{m} a'$ **in A**, fm is just $f(a) \xrightarrow{fm} f(b)$, similarly for g. (They preserve domains, codomains identity and composition). Moreover, for any $a \in \mathcal{A}, b \in \mathcal{B}$,

$$\mathcal{B}(fa, b) = \{fa \leq b\} \iff f(a) \leq b \iff a \leq g(b) \iff \mathcal{A}(a, gb) = \{a \leq gb\},$$
$$\text{and } \mathcal{B}(fa, b) = \emptyset \iff f(a) \not\leq b \iff a \not\leq g(b) \iff \mathcal{A}(a, gb) = \emptyset.$$

So, there is a bijection between $\mathcal{B}(fa, b) \cong \mathcal{A}(a, gb)$. We also get the naturality because their home-sets contain always only one element if it is not empty, just need to make sure the no-emptiness which is ensured by transitivity. Thus the functor f is the left adjoint to the functor g.

2.7.6 Proposition. Let $\mathcal{A} \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathcal{B}$ be a pair of functors in opposite directions. Saying that $\mathbb{F}$ is a left adjoint to $\mathbb{G}$ is equivalent that saying there exists a natural transformation $\eta : Id_{\mathcal{A}} \overset{\bullet}{\Longrightarrow} \mathbb{G}\mathbb{F}$ such that:

$$\forall A \in \mathcal{A}, B \in \mathcal{B} : \quad \forall A \xrightarrow{h} \mathbb{G}B, \exists! g : \mathbb{F}A \longrightarrow B : \mathbb{G}g \circ \eta_A = h. \tag{2.7.5}$$

Proof. We assume (2.7.1), and for any $\mathcal{A}$-object A, $\eta_A := \varepsilon_{A,\mathbb{F}A}(Id_{\mathbb{F}A})$. Let $A' \xrightarrow{m} A$ ***in*** $\mathcal{A}$. If we put $m' = \mathbb{F}m$, $f = Id_{\mathbb{F}A}$ and $f' = Id_{\mathbb{F}A'}$ in the equations (2.7.2) and (2.7.3), we get:

$$\varepsilon_{A,\mathbb{F}A}(Id_{\mathbb{F}A}) \circ m = \varepsilon_{A',\mathbb{F}A}(\mathbb{F}m) = \mathbb{G}\mathbb{F}m \circ \varepsilon_{A',\mathbb{F}A'}(Id_{\mathbb{F}A'}).$$

Thus the naturality of $\eta = (\eta_A)_{A \in \mathcal{A}} : Id_{\mathcal{A}} \overset{\bullet}{\Longrightarrow} \mathbb{G}\mathbb{F}$. Now, let $A \xrightarrow{h} \mathbb{G}B$ in $\mathcal{A}$. Taking $g = \varepsilon^{-1}_{A,B}(h)$,

$$\mathbb{G}g \circ \eta_A = \mathbb{G}g \circ \varepsilon_{A,\mathbb{F}A}(Id_{\mathbb{F}A}) \overset{(2.7.3)}{=} \varepsilon_{A,B}(g) = h.$$

If $\mathbb{F}A \xrightarrow{g'} B$ satisfies $\mathbb{G}g' \circ \eta_A = h$, $h = \mathbb{G}g' \circ \eta_A = \mathbb{G}g' \circ \varepsilon_{A,\mathbb{F}A}(Id_{\mathbb{F}A}) \overset{(2.7.3)}{=} \varepsilon_{A,B}(g')$.
So, $g' = \varepsilon^{-1}_{A,B}(h) = g$. Thus the existence and uniqueness for (2.7.5) are well-satisfied.
Conversely, for any $\mathcal{A}$-object A, $\mathcal{B}$-object B and $\mathbb{F}A \xrightarrow{g} B$ in $\mathcal{B}$, we set $\varepsilon_{A,B}(g) = \mathbb{G}g \circ \eta_A$. By (2.7.5), $\varepsilon_{A,B}$ is a bijection. Moreover, for $A' \xrightarrow{m} A$ in $\mathcal{A}$, $B \xrightarrow{m'} B'$ in $\mathcal{A}$ and $f, f' \in \mathcal{B}(\mathbb{F}A, B)$, we get

$$\varepsilon_{A,B}(f) \circ m = \mathbb{G}f \circ \eta_A \circ m \overset{\text{naturality}}{=} \mathbb{G}f \circ \mathbb{G}\mathbb{F}m \circ \eta_{A'} = \mathbb{G}(f \circ \mathbb{F}m) \circ \eta_{A'} = \varepsilon_{A',B}(f \circ \mathbb{F}m), \text{ and}$$
$$\mathbb{G}m' \circ \varepsilon_{A,B}(f') = \mathbb{G}m' \circ \mathbb{G}f' \circ \eta_A = \mathbb{G}(m' \circ f') \circ \eta_A = \varepsilon_{A,B'}(m \circ f').$$

Then we get the isomorphism $(\varepsilon_{A,B})_{A \in \mathcal{A}, B \in \mathcal{B}}$ that is natural both in A and in B. Hence (2.7.1). □

2.7.7 Remark. These propositions are also equivalent to say: there exists a natural $\vartheta : \mathbb{F}\mathbb{G} \overset{\bullet}{\Longrightarrow} Id_{\mathcal{B}}$ such that for any $\mathcal{A}$-object A and $\mathcal{B}$-object B,

$$\begin{tikzcd} \mathbb{F}A \arrow[d, "\forall m \text{ in } \mathcal{B}"'] & A \arrow[d, dashed, "\exists! p"'] & \mathbb{F}A \arrow[d, "\mathbb{F}(p)"'] \arrow[dr, "m"] & \\ B & \mathbb{G}B & \mathbb{F}\mathbb{G}B \arrow[r, "\vartheta_B"'] & B \end{tikzcd} \quad \text{commutes.}$$

Here, the relations of ϑ and the bijection ε are $\vartheta_B = \varepsilon^{-1}_{\mathbb{G}B,B}(Id_{\mathbb{G}B})$ and $\varepsilon^{-1}_{A,B} = \vartheta_B \circ \mathbb{F}$. We could prove these using the duality of the proof above. In these cases, the natural transformation η is said to be the **unit** of the adjunction and the natural transformation ϑ is said to be the **counit** of the adjunction.

2.7.8 Example. For an object $L \in \mathbf{Loc}$, we denote $CPF(L)$ the collection of all complete prime filters in L, $\Sigma_a = \{F \in CPF(L)/a \in F\}$ for $a \in L$ and the spectrum

$$Sp(L) = \Big(CPF(L), \{\Sigma_a/a \in L\}\Big).$$

It is known (see Picado and Pultr (2011)) that $Sp(L)$ is a topological space with toplogy $(\Sigma_a)_{a\in L}$.
If $f : L \longrightarrow M$ is a localic map then it is a right adjoint to a $\mathbf{Frm}$-morphism f^*. Let's define $Sp(f) : Sp(L) \to Sp(M)$, $F \mapsto Sp(f)(F) = (f^*)^{-1}[F]$ which is well-defined because a preimage of a compete prime filter by a frame homomorphism, is a complete prime filter. It is clear that $(Sp(f))^{-1}[\Sigma_a] = \Sigma_{f^*(a)}$ then $Sp(f)$ is continuous map. Moreover it is clear that $Sp(Id_L) = Id_{Sp(L)}$ and $Sp(g \circ f) = Sp(g) \circ Sp(f)$. Thus we have a functor $Sp : \mathbf{Loc} \longrightarrow \mathbf{Top}$.

$$\begin{aligned} \mathbf{Sp} : \mathbf{Loc} &\to \mathbf{Top} \\ L &\mapsto \mathbf{Sp}(L) = Sp(L) \\ L \xrightarrow{f} M &\mapsto \mathbf{Sp}(f) = Sp(f). \end{aligned}$$

Moreover, if X is $\mathbf{Top}$-object, the map $(X, \tau_X) \xrightarrow{\lambda_X} \mathbf{Sp(Lc(}X\mathbf{))}, x \mapsto \mathcal{V}(x) = \{U \in \tau_X / x \in U\}$ is well-defined because if $U, V \in \mathcal{V}(x)$ and $U \subseteq W$ then $U \cap V, W \in \mathcal{V}(x)$ and if $\cup_{i\in I} U_i \in \mathcal{V}(x)$, $x \in U_i$ for some $i \in I$ and $U_i \in \mathcal{V}(x)$ for some $i \in I$.

In addition, it is continuous maps because for any $O \in \mathbf{Lc}(X)$, $\lambda_X^{-1}(\Sigma_O) = \{x \in X : \mathcal{V}(x) \in \Sigma_O\} = \{x : O \in \mathcal{V}(x)\} = \{x : x \in O\} = O \in \tau_X$. We get a natural transformation $\lambda : Id_{\mathbf{Top}} \overset{\bullet}{\Longrightarrow} \mathbf{SpLc}$. Indeed, as we see at Example 2.3.2 , $\mathbf{Lc}(f) = (\Omega(f))_*$, $\mathbf{Sp}(\mathbf{Lc}(f)) = (\Omega(f))^{-1} = (f^{-1})^{-1}$ for any $X \xrightarrow{f} Y$ $\mathbf{Top}$-morphism. Combining this fact with $\{O : f^{-1}[O] \in \mathcal{V}(x)\} = \{O : x \in f^{-1}[O]\} = \mathcal{V}(f(x))$ for all $x \in X$, we get $\lambda_Y \circ f = \mathbf{Sp}(\mathbf{Lc}(f)) \circ \lambda_X$.
The universal property is by Theorem 4.6.3 in Picado and Pultr (2011) page 17. So we get $\mathbf{Lc} \dashv \mathbf{Sp}$.

2.7.9 Remark. If $\mathcal{A} \underset{\mathbb{G}}{\overset{\mathbb{F}}{\rightleftarrows}} \mathcal{B} \underset{\mathbb{J}}{\overset{\mathbb{H}}{\rightleftarrows}} \mathcal{C}$ be functors such that $\mathbb{F} \dashv \mathbb{G}$ and $\mathbb{H} \dashv \mathbb{J}$, then we have $\mathbb{H}\mathbb{F} \dashv \mathbb{G}\mathbb{J}$.

2.7.10 Theorem (RAPL,LAPC Picado and Pultr (2011))**.** Right adjoints preserve limits and left adjoints preserve colomits. More precisely, for an adjunction $\mathcal{A} \underset{\mathbb{G}}{\overset{\mathbb{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{B}$, if $(f_a : X \longrightarrow \mathbb{D}(a))_{a\in\mathcal{K}}$ is a limit of a diagram $\mathbb{D} : \mathcal{K} \to \mathcal{B}$ then $(\mathbb{G}f_a : \mathbb{G}X \longrightarrow \mathbb{G}\mathbb{D}(a))_{a\in\mathcal{K}}$ is a limit of a diagram $\mathbb{G}\mathbb{D} : \mathcal{K} \to \mathcal{A}$ and similarly for $\mathbb{F}$ and colimits (duality).

Proof. For any $a \xrightarrow{m} b$ in $\mathcal{K}$, $\mathbb{GD}(m) \circ \mathbb{G}f_a = \mathbb{G}((\mathbb{D}m)f_a) = \mathbb{G}f_b$. Then $(\mathbb{G}f_a : \mathbb{G}X \longrightarrow \mathbb{GD}(a))_{a\in\mathcal{K}}$ is lower bound of $\mathbb{GD}$. Let $(g_a : Y \to \mathbb{GD}(a))_{a\in\mathcal{K}}$ a lower bound of $\mathbb{GD}$. Recall from Remark 2.7.7 that $\varepsilon^{-1}_{Y,\mathbb{D}a}(g_a) = \vartheta_{\mathbb{D}a} \circ \mathbb{F}(g_a) : \mathbb{F}Y \to \mathbb{D}a$ where $\vartheta : \mathbb{FG} \overset{\bullet}{\Longrightarrow} Id_{\mathcal{C}B}$ is the counit of the adjunction.

For any $a \xrightarrow{m} b$ in $\mathcal{K}$,

$$\mathbb{D}m \circ \varepsilon^{-1}_{Y,\mathbb{D}a}(g_a) = \mathbb{D}m \circ \vartheta_{\mathbb{D}a} \circ \mathbb{F}(g_a) = \vartheta_{\mathbb{D}b} \circ \mathbb{FGD}m \circ \mathbb{F}g_a = \vartheta_{\mathbb{D}b} \circ \mathbb{F}(\mathbb{GD}a \circ g_a) = \vartheta_{\mathbb{D}b} \circ \mathbb{F}g_b = \varepsilon^{-1}_{Y,\mathbb{D}b}(g_b).$$

Then $(\varepsilon^{-1}_{Y,\mathbb{D}a}(g_a) : \mathbb{F}Y \to \mathbb{D}a)_{a\in\mathcal{K}}$ is lower bound of $\mathbb{D}$. Thus there is an unique morphism in $\mathcal{B}$ from $\mathbb{F}Y$ to X, h such that $f_a \circ h = \varepsilon^{-1}_{Y,\mathbb{D}a}(g_a)$ for every $a \in \mathcal{K}$. Then for any $a \in \mathcal{K}$,

$$g_a = \varepsilon(f_a h) = \mathbb{G}(f_a h) \circ \eta_Y = \mathbb{G}f_a \circ \mathbb{G}h \circ \eta_Y = \mathbb{G}f_a \circ \varepsilon(h).$$

Then $\varepsilon_{Y,\mathbb{D}a}(h)$ is a morphism in $\mathcal{A}$ from Y to $\mathbb{G}X$ satisfying $g_a = \mathbb{G}f_a \circ \varepsilon(h)$.
Let f be a morphism in $\mathcal{A}$ from Y to $\mathbb{F}X$ satisfying $g_a = \mathbb{G}f_a \circ f$ for any $a \in \mathcal{K}$. Then

$$\begin{aligned}
\varepsilon^{-1}_{Y,\mathbb{D}a}(g_a) &= \varepsilon^{-1}_{Y,\mathbb{D}a}\Big(\mathbb{G}f_a \circ \varepsilon_{Y,\mathbb{D}a}(\varepsilon^{-1}_{Y,\mathbb{D}a}(f))\Big) &&= \varepsilon^{-1}_{Y,\mathbb{D}a}\Big(\mathbb{G}f_a \circ \mathbb{G}(\varepsilon^{-1}_{Y,\mathbb{D}a}(f)) \circ \eta_Y\Big) = \varepsilon^{-1}_{Y,\mathbb{D}a}\Big(\mathbb{G}(f_a\varepsilon^{-1}_{Y,\mathbb{D}a}(f)) \circ \eta_Y\Big)\\
&= \varepsilon^{-1}_{Y,\mathbb{D}a}\Big(\varepsilon_{Y,\mathbb{D}a}(f_a\varepsilon^{-1}_{Y,\mathbb{D}a}(f))\Big) &&= f_a\varepsilon^{-1}_{Y,\mathbb{D}a}(f).
\end{aligned}$$

Thus $\varepsilon^{-1}_{Y,\mathbb{D}a}(f)$ satisfies $f_a \circ \varepsilon^{-1}_{Y,\mathbb{D}a}(f) = \varepsilon^{-1}_{Y,\mathbb{D}a}(g_a)$ for every $a \in \mathcal{K}$. By uniqueness of such h, $\varepsilon^{-1}_{Y,\mathbb{D}a}(f) = h$. Therefore $f = \varepsilon_{Y,\mathbb{D}a}(h)$. Hence the uniqueness of the morphism $\varepsilon_{Y,\mathbb{D}a}(h)$ satisfying $g_a = \mathbb{G}f_a \circ \varepsilon_{Y,\mathbb{D}a}(h)$. In conclusion, $(\mathbb{G}f_a : \mathbb{G}X \longrightarrow \mathbb{GD}(a))_{a\in\mathcal{K}}$ is a limit of a diagram $\mathbb{GD}$. □

The following theorems are found in Riehl (2017). We state them here without providing the proofs.

2.7.11 Theorem (General Adjoint Functor Theorem)**.** Let $\mathcal{A}$ a locally small and complete category. Let a limit-preserving functor $\mathbb{F} : \mathcal{A} \longrightarrow \mathcal{C}$ that satisfies the following **solution set condition** :

$$\forall O \in \mathcal{C}, \exists S_O = \{O \xrightarrow{m_i} \mathbb{F}A_i\} \text{ small} : \Big(\forall O \xrightarrow{m} \mathbb{F}A, \exists m_i \in S_O, A_i \xrightarrow{f} A \textit{ in } \mathcal{A} : m = (\mathbb{F}f) \circ m_i\Big). \tag{2.7.6}$$

Then $\mathbb{F}$ admits a left adjoint.

2.7.12 Example. Let $(A, \leq), (B, \leq)$ be two complete lattices and $f : (A, \leq) \to (B, \leq)$ a infinima-preserving (all meets). In Remark 2.7.5 and Example 2.6.6, we can viewed f as a functor between the categories $(A, \leq), (B, \leq)$ complete lattice means they are complete category and infinima-preserving means limit-preserving. Moreover for any $b \in B$, we can take $S_b = \{b \leq f(x) : x \in A\}$ which is a small set and verifies for any $m =$"$b \leq f(a)$" in $\leq_B (b, f(a))$ there is $m_a =$"$b \leq f(a)$" $\in S_b$ and $Id_a = a \leq a$ in $\leq_A$ such that $m = f(Id_a) \circ m_a$. So the solution set condition is satisfied.

Thus f admits a left adjoint which is also the left Galois adjoint.

This example is the converse of Remark 2.2.11 but Remark 2.2.11 is an immediate consequence of Theorem 2.7.10 with Examples 2.6.6, 2.7.5.

2.7.13 Proposition (Riehl (2017) Propisition 4.4.1)**.** If $\mathbb{F}$ and $\mathbb{F}'$ are left adjoints to G, then $\mathbb{F} \cong \mathbb{F}'$, and moreover there is an unique natural isomorphism $\theta : \mathbb{F} \cong \mathbb{F}'$ commuting with the units and counits of the adjunctions: $\eta' = (\mathbb{G}\theta) \circ \eta$ and $\vartheta = (\theta\mathbb{G}) \circ \vartheta$.

3. Free Objects

In this part, we abuse the forgetful functor, sometimes, we won't define the domain, the codomain, the arrow-mapping, and object mapping. But, the notation $\mathbb{U}$ is considered for the forgetful functor unless stated otherwise.
In this project, we consider $0 \in \mathbb{N}$ and we denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

3.1 Free objects and free functors

3.1.1 Definition. Given a forgetful functor $\mathbb{U} : \mathcal{D} \longrightarrow \mathcal{C}$ and A a $\mathcal{C}$-object. The $\mathcal{D}$-object FA is said to be a free $\mathcal{D}$-object on A with respect to $\mathbb{U}$ if there is a $\mathcal{C}$-morphism $A \xrightarrow{\mu_A} \mathbb{U}FA$ such that for every $\mathcal{D}$-object D and $\mathcal{C}$-morphism $A \xrightarrow{f} \mathbb{U}D$, there exists an unique $\mathcal{D}$-morphism $FA \xrightarrow{g} D$ such that $\mathbb{U}(g) \circ \mu_A = f$.

3.1.2 Example. Consider the forgetful functor $\mathbb{U} : \mathbf{Mon} \longrightarrow \mathbf{Set}$ and $X = \{x\}$ a $\mathbf{Set}$-object. $FX = \{x^n : n \in \mathbb{N}\}$ is a monoid if we consider the binary operation $\star$ with $x^n \star x^m = x^{x+m}$. It is the free $\mathbf{Mon}$-object on X because we have a map $X \xrightarrow{\mu} \mathbb{U}FX$, $x \mapsto x$, satisfying for every monoid M and map $X \xrightarrow{f_a} \mathbb{U}M$, $x \mapsto a$, there is an unique monoid homomorphism $FX \xrightarrow{g_a} M$, $x^n \mapsto a^n$ so that $\mathbb{U}(g_a) \circ \mu = f_a$. In particular, if $x = 1$ and we think that the operation is $+$, the free monoid generated by $\{1\}$ is $(\mathbb{N}, +)$ (ie $1^n = n$), but otherwise, a free monoid with one generator is isomorphic to $(\mathbb{N}, +)$.

3.1.3 Remark. If $\mathbb{U}$ has a left adjoint $\mathbb{F}$, the free $\mathcal{D}$-object on A respect to $\mathbb{U}$ is none other than $\mathbb{F}A$, with the $\mathcal{C}$-morphism, $A \xrightarrow{\mu_A} \mathbb{U}\mathbb{F}A$, the component of the unit of the adjuction.
In this case, we call $\mathbb{F}$: free functor respect to $\mathbb{U}$ (or simply free functor of $\mathbb{U}$). So, if $\mathbb{U}$ admits a left adjoint, we get the free object. Now, we ask if the non-existence of free functor implies the non-existence of free object. We give some response at Theorem 3.1.5.

3.1.4 Free Objects in $\mathbf{Grp}$: free group.

For any set S, we define:

$$FS := \{\prod_{i=1}^{n} z_i : n \in \mathbb{N}, \quad \forall i, z_i \in S \text{ or } z_i = s_i^{-1}, \quad s_i \in S\}. \tag{3.1.1}$$

Here $\prod_{i=1}^{n} z_i = z_1 \cdot z_2 \cdots z_n$ (the order matters). We assume the existence of empty word $e = \prod_{s \in \emptyset} s$. Equipping FS with the binary operation concatenation such that for all $s, u, v \in S : ss^{-1} = s^{-1}s = e$ and $(su)v = s(uv)$, FS is a group. Next, for any map $f : S \to R$, let denote $\bar{f}(s) = f(s)$ and $\bar{f}(s^{-1}) = (f(s))^{-1}$ for$s \in S$ and define the map:

$$\begin{aligned} Ff : &FS \to FR \\ &\prod_{i=1}^{n} z_i \mapsto \prod_{i=1}^{n} \bar{f}(z_i). \end{aligned}$$

This map is well-defined and it is a group homomorphism by construction. Now, we will check that the functor $\mathbb{G} : \mathbf{Set} \to \mathbf{Grp}$ which sends any set S to FS and any map f to Ff is free functor respect to the forgetful $\mathbb{U} : \mathbf{Grp} \to \mathbf{Set}$. The functoriality is ensured by the fact for any $S \xrightarrow{f} R \xrightarrow{g} T$ in $\mathbf{Set}$, for each $s \in S$, $\overline{g \circ f}(s) = g \circ f(s) = \bar{g} \circ \bar{f}(s)$ and $\overline{g \circ f}(s^{-1}) = (g \circ f(s))^{-1} = \bar{g}((f(s))^{-1}) = \bar{g} \circ \bar{f}(s^{-1})$.

$$\text{Then, } \mathbb{G}(g \circ f)\Big(\prod_{i=1}^{n} z_i\Big) = \prod_{i=1}^{n} \overline{g \circ f}(z_i) = \prod_{i=1}^{n} \bar{g} \circ \bar{f}(z_i) = \mathbb{G}g\Big(\prod_{i=1}^{n} \bar{f}(z_i)\Big) = \mathbb{G}g \circ \mathbb{G}f\Big(\prod_{i=1}^{n} z_i\Big).$$

The collection of maps $\eta = (\eta_S : S \to \mathbb{UG}S, x \mapsto x)_{S \in \mathbf{Set}}$ defines clearly a natural transformation $\eta : Id_{\mathbf{Set}} \overset{\bullet}{\Longrightarrow} \mathbb{UG}$. Moreover, for any set S, group G and map $f : S \longrightarrow \mathbb{U}G$, there is an unique group homomorphism $g : \mathbb{G}S \to G, \prod_{i=1}^{n} z_i \mapsto \prod_{i=1}^{n} \bar{f}(z_i)$ so that $\mathbb{U}(g) \circ \eta_S = f$.
Thus, $\mathbb{G}$ is a free functor respect to $\mathbb{U}$.

3.1.5 Theorem. The forgetful functor $\mathbb{U} : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint (free functor) if and only if each element admits a free $\mathcal{D}$-object.

Proof. We follow the notation in Definition 3.1.1. From the right to left, let $A \xrightarrow{m} B$ in $\mathcal{C}$. We get FB a $\mathcal{D}$-object and $\mu_B \circ m : A \to \mathbb{U}FB$ a $\mathcal{C}$-morphism. Then, there exists an unique morphism in $\mathcal{D}(FA, FB)$, (we call it Fm), such that $\mathbb{U}(Fm) \circ u_A = \mu_B \circ m$. So the correspondance

$$\begin{aligned} \mathbb{F} : \mathcal{C} &\to \mathcal{D} \\ A &\mapsto \mathbb{F}(A) = FA \\ A \xrightarrow{m} B &\mapsto \mathbb{F}(m) = Fm \end{aligned}$$

is well-defined. Now, let $A \xrightarrow{m} B \xrightarrow{n} C$. Focusing only on m, we have $\mathbb{U}(\mathbb{F}m) \circ u_A = \mu_B \circ m$ and only on n we have $\mathbb{U}(\mathbb{F}n) \circ u_B = \mu_C \circ n$. Then, we get $\mathbb{F}n \circ \mathbb{F}m : \mathbb{F}A \to \mathbb{F}C$ satisfying

$$\mathbb{U}(\mathbb{F}n \circ \mathbb{F}m) \circ \mu_A = \mathbb{U}(\mathbb{F}n) \circ \mathbb{U}(\mathbb{F}m) \circ \mu_A = \mathbb{U}(\mathbb{F}n) \circ \mu_B \circ n = \mu_C \circ n \circ m.$$

By uniqueness of $F(n \circ m)$ in $\mathcal{D}(FA, FC)$ so that $\mathbb{U}(F(n \circ m)) \circ \mu_A = \mu_C \circ n \circ m$, $\mathbb{F}(n \circ m) = \mathbb{F}n \circ \mathbb{F}m$. Thus, $\mathbb{F}$ is a functor. The condition $\mathbb{U}(\mathbb{F}m) \circ u_A = \mu_B \circ m$ (for any $A \xrightarrow{m} B$ in $\mathcal{C}$) ensures us the naturality of $\mu = (\mu_A)_{A \in \mathcal{C}} : Id_{\mathcal{C}} \overset{\bullet}{\Longrightarrow} \mathbb{UF}$. The universal property is given by the definition of free objects. So, the functor $\mathbb{F}$ is the left adjoint to the forgetful functor $\mathbb{U}$.

The left to right one is at Remark 3.1.3. □

3.1.6 Remark. Let $\mathbb{U} : \mathcal{D} \to \mathcal{C}$ be a forgetful functor, A be a $\mathcal{C}$-object, and F be free $\mathcal{D}$-object on A respect to $\mathbb{U}$. For any D $\mathcal{D}$-object, D is also a free object on A if and only if D and F are isomorphic.

Proof. From left to right, let $\mu_F : A \to \mathbb{U}F$ and $\mu_D : A \to \mathbb{U}D$ so that F and D are free objects on A. Then there is an unique f ***in*** $\mathcal{F}$ such that $\mu_D = \mathbb{U}(f) \circ \mu_D$ and an unique g ***in*** $\mathcal{D}$ such that $\mu_F = \mathbb{U}(g) \circ \mu_D$. Hence $\mu_D = \mathbb{U}(fg) \circ \mu_D$ and $\mu_F = \mathbb{U}(gf) \circ \mu_F$. By definition of free object, Id_F is the unique $\mathcal{D}$-morphism such that $\mu_D = \mathbb{U}(Id_D) \circ \mu_D$. Then $fg = Id_D$. Similarly for $Id_F = gf$. Hence $F \cong D$. For the another direction, $\mu_F : A \to \mathbb{U}F$ is already given and $F \underset{g}{\overset{f}{\rightleftarrows}} D$ be morphisms isomorphic. We can easily check that $\mu_D = \mathbb{U}(f) \circ \mu_F$ allows us to conclude that D is a free $\mathcal{D}$ object on A. □

3.2 Free metric space and completion of metric

Recall that the category of metric spaces $\mathbf{Met}$ has objects that are metric spaces and morphisms that are continuous maps. More precisely the objects of $\mathbf{Met}$ are the pairs (S, d) where S be a set and

d is map from $S \times S$ to $\mathbb{R}$ such that $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles), $d(x, y) = d(y, x)$ (symmetry) and $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality) for any $x, y, z \in S$. The $\mathbf{Met}$-morphisms from (S, d) to (R, e) are the maps $f : S \longrightarrow R$ such that for any $a \in S, \varepsilon > 0$ there is $\alpha > 0$ such that for all $x \in S$, $d(x, a) < \alpha$ implies $e(f(x), f(a)) < \varepsilon$.
In this part, let's think about how to get the left adjoint to the forgetful $\mathbb{U} : \mathbf{Met} \to \mathbf{Set}$. The trivial way is for any set S, we equip it with the discrete metric $d_S : S \times S \to \mathbb{R}$ which assigns (x, y) to 0 if $x = y$ and to 1 otherwise. d_S is known as metric.

3.2.1 Lemma. Let (R, e) be a metric space and $f : S \longrightarrow R$ be any map. Epuipping S with the discrete metric d_S that we have talked above, $g : (S, d_S) \longrightarrow (R, e), x \mapsto f(x)$ is a $\mathbf{Met}$-morphism.

Proof. Let $a \in S$ and $\varepsilon > 0$. We can take $\alpha = 0.5$. It verifies that for any $x \in S$: if $d_S(x, a) < \alpha$ which implies $d(x, a) = 0$ ($x = a$ and $f(x) = f(a)$) then $e(f(x), f(a)) = 0 < \varepsilon$. □

Now, let's construct the functor $\mathbb{M} : \mathbf{Set} \to \mathbf{Met}$ that assigns each set S to the metric space $\mathbb{M}S = (S, d_S)$ and each map $f : S \longrightarrow R$ to $\mathbb{M}f : (S, d_S) \longrightarrow (R, d_R), x \mapsto f(x)$ which is a $\mathbf{Met}$-morphism by Lemma 3.2.1. Obviously, it respects composition, identity.
By noting that $\mathbb{UM} = Id_{\mathbf{Set}}$, we can take the natural transformation $\eta : Id_{\mathbf{Set}} \overset{\bullet}{\Longrightarrow} \mathbb{UM}$ where the components are the identity maps for each set.The universal property is also ensured by Lemma 3.2.1. Hence $\mathbb{M}$ is the free functor respect to the forgetful $\mathbb{U} : \mathbf{Met} \to \mathbf{Set}$.

On the other hand, if we consider the category $\mathbf{Met}_u$ of metric spaces where morphisms are the uniformly continuous maps. We recall that the map $f : (S, d) \longrightarrow (R, d_S)$ is uniformly continuous if for any $\varepsilon > 0$ there is $\alpha > 0$ such that for all $a, x \in S$, $d(x, a) < \alpha$ implies $e(f(x), f(a)) < \varepsilon$.
We can repeat all of the computations above and all things work well. So we also get the functor $\mathbb{M}_c : \mathbf{Set} \to \mathbf{Met}_u$ left adjoint to the forgetful $\mathbb{U}_u : \mathbf{Met}_u \to \mathbf{Set}$.

3.2.2 Definition. Let (S, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. $(x_n)_{n\in\mathbb{N}}$ is said to be Cauchy sequence if

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \quad (\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon).$$

Recall that $(x_n)_{n\in\mathbb{N}}$ is convergent vers a means for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ satisfying for all $n \geq n_0$ $d(x_n, a) < \varepsilon$. We can easily check that convergent implies Cauchy but the converse is not true in general. For instance, we know that $x_n = (1 + \frac{1}{n})^n \in \mathbb{Q} \subseteq \mathbb{R}$ for any $n \in \mathbb{N}$ and it converges to e. It is convergent in $\mathbb{R}$ then Cauchy in $\mathbb{R}$. Hence it is Cauchy in $\mathbb{Q}$ but not convergent in $\mathbb{Q}$ as its limit $e \notin \mathbb{Q}$.

3.2.3 Definition. A metric space (S, d) is said to be complete if every Cauchy sequence in (S, d) is convergent in (S, d).

3.2.4 Proposition. Let (S, d) be a metric space and $\mathcal{S}$ be the collection of all Cauchy sequences in (S, d). We define $\sim$ a binary relation on $\mathcal{S}$ defined by $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if and only if $\left(d(x_n, y_n)\right)_{n\in\mathbb{N}}$ converge to zero in $\mathbb{R}$ equipped the trivial metric[1] $|\cdot - \cdot|$. Then,

(a) $\sim$ is an equivalence relation on $\mathcal{S}$. Let's denote $\tilde{S} = \mathcal{S}/\sim$ and write $\tilde{x}$ the class of $(x_n)_{n\in\mathbb{N}}$;

(b) for any sequence $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$, $\left(d(x_n, y_m)\right)_{n\in\mathbb{N}}$ converges in $(\mathbb{R}, |\cdot - \cdot|)$; and

(c) $\tilde{d}(\tilde{x}, \tilde{y}) = \lim d(x_n, y_n)$ is metric on $\tilde{S}$ and $(\tilde{S}, \tilde{d})$ is complete.

[1] We recall that the trivial metric in $\mathbb{R}$, $d(x, y) = |x - y|$.

Proof. For (a) the reflexivity is from the identity of indiscernibles axiom of metric, the symetricity is from the symetricity axiom of metric and the transitivity is from the triangle inequality and the Sandwich Theorem. For (b), using the triangle inequality and property of absolute value we have

$$\forall n, m \in \mathbb{N}: \quad |d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m). \tag{3.2.1}$$

Combining this with the fact that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are in $\mathcal{S}$, the sequence $\left(d(x_n, y_n)\right)_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{R}, |\cdot - \cdot|)$. Then $\left(d(x_n, y_m)\right)_{n\in\mathbb{N}}$ is convergent (because we know that is $(\mathbb{R}, |\cdot - \cdot|)$ complete). For (c), to prove that $\tilde{d}$ is a metric, it remains only to check that it is well-defined as a map. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ and $(y'_n)_{n\in\mathbb{N}}$ be in $\mathcal{S}$ such that $(x_n)_{n\in\mathbb{N}} \sim (x'_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}} \sim (y'_n)_{n\in\mathbb{N}}$. We need to prove that $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x'}, \tilde{y'})$, we use the same technique as in (3.2.1) to get

$$\forall n, m \in \mathbb{N}: \quad |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n).$$

Combining this with the fact that $(x_n)_{n\in\mathbb{N}} \sim (x'_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \sim (y'_n)_{n\in\mathbb{N}}$ with the Sandwich Theorem, we have $d(x_n, y_n) - d(x'_n, y'_n)$ converge to zero which implies $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}', \tilde{y}')$.

Now, we should prove that the metric space $(\tilde{S}, \tilde{d})$ is complete. Let $(X^{(k)})_{k\in\mathbb{N}} = \left(\widetilde{(x_n^{(k)})_{n\in\mathbb{N}}}\right)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\tilde{S}, \tilde{d})$. Then

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N}: \quad \Big(\forall k, l \geq K_\varepsilon: \quad \tilde{d}(X^{(k)}, X^{(l)}) = \lim_{j\to\infty} d(x_j^{(k)}, x_j^{(l)}) < \frac{\varepsilon}{2}\Big). \tag{3.2.2}$$

Moreover, for any $k \in \mathbb{N}$, the sequence $X^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$ (one representative of the class) is Cauchy in (S, d). Then

$$\forall k \in \mathbb{N}, \exists a_k \in \mathbb{N}: \quad \Big(\forall j > a_k: \quad d(x_{a_k}^{(k)}, x_j^{(k)}) < \frac{1}{k}\Big). \tag{3.2.3}$$

Next, $X := (x_{a_n}^n)_{n\in\mathbb{N}}$ which is by definition a sequence in S. Let's show that it is Cauchy. Combining (3.2.2) and (3.2.3) with the triangle inequality, we can write

$$\forall \varepsilon > 0, \exists K_\varepsilon \in \mathbb{N}: \quad \Big(\forall k, l > K_\varepsilon: d(x_{a_k}^{(k)}, x_{a_l}^{(l)}) \leq \lim_{j\to\infty} \Big(d(x_{a_k}^{(k)}, x_j^{(k)}) + d(x_j^{(k)}, x_j^{(l)}) + d(x_j^{(l)}, x_{a_l}^{(l)})\Big) < \frac{1}{k} + \frac{\varepsilon}{2} + \frac{1}{l}\Big).$$

$$\text{Then } \forall \varepsilon > 0, \exists K = \max(K_\varepsilon, \lceil \frac{5}{\varepsilon} + 1 \rceil) \in \mathbb{N}: \quad \Big(\forall k, l > K: d(x_{a_k}^{(k)}, x_{a_l}^{(l)}) < \varepsilon\Big). \tag{3.2.4}$$

Hence X is a Cauchy sequence in (S, d). We claim that $(X^{(k)})_{k\in\mathbb{N}}$ converges to the class of X in $(\tilde{S}, \tilde{d})$. Using the triangle inequality with (3.2.3) and (3.2.4), we get

$$\forall \varepsilon > 0, \exists K' = \max(K, \lceil \frac{2}{\varepsilon} \rceil) \Big(\forall k > K: \tilde{d}(X^{(k)}, X) = \lim_{n\to\infty} d(x_n^{(k)}, x_n^{(a_n)}) \leq \lim_{n\to\infty} d(x_n^{(k)}, x_{a_k}^{(k)}) + d(x_{a_k}^{(k)}, x_{a_n}^{(n)})\Big) < 2\varepsilon.$$

Thus $(X^{(k)})_{k\in\mathbb{N}}$ converges to X in $(\tilde{S}, \tilde{d})$. Therefore $(\tilde{S}, \tilde{d})$ is complete. □

Therefore, for every metric space (S, d), we can get from it a complete metric space $(\tilde{S}, \tilde{d})$.

3.2.5 Remark. For any $\mathbf{Met}_u$-morphism $(S, d) \xrightarrow{f} (R, e)$, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy implies $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy and $\left(d(x_n, y_n)\right)_{n\in\mathbb{N}}$ converges to zero implies $\left(e(f(x_n), f(y_n))\right)_{n\in\mathbb{N}}$ converges to zero. If in addition, if (R, e) is complete $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ converge to the same limit.

Now, let's construct a functor from $\mathbf{Met}_u$ to $\mathbf{CMet}_u$ category of complete metric spaces whose morphisms are uniformly continuous maps. The proposition above leads us to construct

$$\begin{aligned}\mathbb{C}: \mathbf{Met}_u &\to \mathbf{CMet}_u \\ (S,d) &\mapsto \mathbb{C}(S,d) = (\tilde{S},\tilde{d}) \\ (S,d) \xrightarrow{f} (R,e) &\mapsto (\tilde{S},\tilde{d}) \xrightarrow{\mathbb{C}f} (\tilde{R},\tilde{e}), \tilde{x} \mapsto (\widetilde{f(x_n)})_{n\in\mathbb{N}}.\end{aligned}$$

Remark 3.2.5 tells us that $\mathbb{C}f$ is well-defined as a map. We claim that it is uniformly continuous. Let $\varepsilon > 0$. Because f is uniformly continuous, there is $\alpha > 0$ such that for any $x, y \in S$, $d(x,y) < \alpha$ implies $e(f(x), f(y)) < \frac{\varepsilon}{2}$ $(\star)$. We take $\beta = \frac{\alpha}{2}$ and will prove that for any $\tilde{x}, \tilde{y} \in \tilde{S}$, $\tilde{d}(\tilde{x},\tilde{y}) < \beta$ implies $\tilde{e}((\widetilde{f(x_n)})_{n\in\mathbb{N}}, (\widetilde{f(y_n)})_{n\in\mathbb{N}}) < \varepsilon$. $\tilde{d}(\tilde{x},\tilde{y}) < \beta$ means $\lim d(x_n,y_n) < \frac{\alpha}{2}$. Then we have

$$\begin{aligned}\forall\epsilon > 0, \exists n_\epsilon \in \mathbb{N}: \quad (\forall n \geq n_\epsilon : d(x_n,y_n) < \frac{\alpha}{2} + \epsilon) \quad &\implies \exists n_o \in \mathbb{N}, \forall n \geq n_0 : d(x_n,y_n) \leq \frac{\alpha}{2} < \alpha \\ \overset{(\star)}{\implies} \exists n_o \in \mathbb{N}, \forall n \geq n_0 : e(f(x_n), f(y_n)) < \frac{\varepsilon}{2} \quad &\implies \lim e(f(x_n), f(y_n)) \leq \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Hence $\mathbb{C}f$ is uniformly continuous and the morphism mapping is also well-defined.
Obviously, it respects identity and composition. Hence, we have a functor $\mathbb{C}$. We denote also $\mathbb{I}$ the inclusion $\mathbf{CMet}_u \to \mathbf{Met}_u$.
Now, let's think about the natural transformation $\eta : Id_{\mathbf{Met}_u} \overset{\bullet}{\implies} \mathbb{IC}$ whose the components are $\eta_{(S,d)}$ that we will simply write $\eta_S : (S,d) \to \mathbb{IC}(S,d) = (\tilde{S},\tilde{d})$, $x \mapsto (x)_{n\in\mathbb{N}}$ (constant sequence).

3.2.6 Lemma. Let (C,c) be a complete metric space. For any $\mathbf{Met}_u$-morphism $(S,d) \xrightarrow{f} \mathbb{I}(C,c)$, there is an unique $\mathbf{CMet}_u$-morphism $(\tilde{S},\tilde{d}) \xrightarrow{g} (C,c)$ such that $\mathbb{I}g \circ \eta_S = f$.

Proof. For any $\tilde{x} \in \tilde{S}$, we set $g(\tilde{x}) = \lim f(x_n)$. The last part of Remark 3.2.5 ensures us that g is well-defined. To prove that such g is $\mathbf{CMet}_u$-morphism we use the same technique to proof of $\mathbb{C}f$, just we need to notice that $\lim c(f(x_n), f(y_n)) = c(\lim f(x_n), \lim f(y_n))$[2]. Clearly, it satisfies $\mathbb{I}g \circ \eta_S = f$. Assume there is another $h \in \mathbf{CMet}_u((\tilde{S},\tilde{d}),(C,c))$ satisfying $\mathbb{I}h \circ \eta_S = f$. We should be able to have for any $\tilde{x} \in \tilde{S}$, $h(\tilde{x}) = g(\tilde{x}) = \lim f(x_n)$. To prove it, we will prove that for any $\varepsilon > 0$, $c(\lim f(x_n), h(\tilde{x})) < \varepsilon$. Let $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ be in $\tilde{S}$ and $\varepsilon > 0$. Because g, h are unifomly continuous, there is $\alpha_1, \alpha_2 > 0$ such that for any $n \in \mathbb{N}$ and $z \in S$, $\lim d(x_n, z) < \alpha_1$ implies $c(\lim f(x_n), f(z)) < \frac{\varepsilon}{2}$ and $\lim d(x_n,z) < \alpha_2$ implies $c(h(\tilde{x}), f(z)) < \frac{\varepsilon}{2}$. We can take $\delta = \min(\alpha_1,\alpha_2)$ and have any $n \in \mathbb{N}$ and $z \in S$, $\lim d(x_n,z) < \delta$ implies $c(h(\tilde{x}), \lim f(x_n)) < \varepsilon$ $(\star\star)$.
As $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there is $n_\delta \in \mathbb{N}$ such that for any $n > n_\delta$ (and we take $z = x_{n_\delta}$): $d(x_n,z) < \delta$ then $\lim d(x_n,z) < \delta$. Combining this with $(\star\star)$, we have $c(h(\tilde{x}), \lim f(x_n)) < \varepsilon$. Therefore

$$\forall\varepsilon > 0 : c(h(\tilde{x}), \lim f(x_n)) < \varepsilon \implies c(h(\tilde{x}), \lim f(x_n)) = 0 \implies h(\tilde{x}) = \lim f(x_n) = g(\tilde{x}).$$

Hence the equality $h = g$ and the unicity. □

3.2.7 Corollary. In consequence, $\mathbb{C}$ is the left adjoint to $\mathbb{I}$. Then $\mathbf{CMet}_u$ is a reflective subcategory of $\mathbf{Met}_u$. Finally, we also get a free functor $\mathbb{F} = \mathbb{CM}_u$ of the $\mathbb{U}_u I : \mathbf{Set} \to \mathbf{CMet}_u$.

Proof. The only thing to check is the fullness of subcategory $\mathbf{CMet}_u$ in $\mathbf{Met}_u$. It is full because there is no additional property for a $\mathbf{Met}_u$-morphism to be a $\mathbf{CMet}_u$-morphism, just the domain and codomain should be $\mathbf{CMet}_u$-objects. □

[2] It is because the metric is continuous then preserves limits.

3.2.8 Some classics free objects (free functors)

In this part, we do not give the details by the condition of pages but we just state the free functors.

(a) Free monoid on set:

$$\begin{aligned}\mathbb{M} : \mathbf{Set} &\to \mathbf{Mon} \\ S \mapsto \mathbb{M}S &= \{\prod_{i=1}^{n} s_i : n \in \mathbb{N}, \quad \forall i : s_i \in S\} \\ S \xrightarrow{f} R \mapsto \mathbb{M}S &\xrightarrow{\mathbb{M}f} \mathbb{M}R, \prod_{i=1}^{n} s_i \mapsto \prod_{i=1}^{n} f(s_i).\end{aligned} \tag{3.2.5}$$

(b) Free R-module on set:

$$\begin{aligned}&\mathbb{M}_R : \mathbf{Set} \to \mathbf{Mod}_R \\ &S \mapsto \mathbb{M}_R S = \{\sum_{s\in S} a_s s : \quad (\forall s \in S : a_s \in R), (\exists X \subseteq S, |X| < \infty : \forall s \in S \setminus X : a_s = 0)\} \\ &S \xrightarrow{f} T \mapsto \mathbb{M}_R S \xrightarrow{\mathbb{M}_R f} \mathbb{M}_R T, \sum_{s\in S} a_s s \mapsto \sum_{s\in S} a_s f(s).\end{aligned} \tag{3.2.6}$$

(c) Free space on set:

$$\begin{aligned}\mathbb{T} : \mathbf{Set} &\to \mathbf{Top} \\ S \mapsto \mathbb{T}S &= (S, \mathcal{P}(S)) \\ S \xrightarrow{f} R \mapsto \mathbb{T}S &\xrightarrow{\mathbb{T}f} \mathbb{T}R, x \mapsto f(x).\end{aligned} \tag{3.2.7}$$

(d) Free abelian group on set: it can be viewed as a composition of free functor at 3.1.4 with the reflector in Example 2.7.2. More precisely, we go from set to group and after from group to abelian group.

(e) Free Ring(unitary) on set:

$$\begin{aligned}\mathbb{R} : \mathbf{Set} &\to \mathbf{Ring} \\ X \mapsto \mathbb{R}X &= \mathbb{Z}[X] \\ X \xrightarrow{f} Y \mapsto \mathbb{R}X &\xrightarrow{\mathbb{R}f} \mathbb{R}Y, x \mapsto f(x).\end{aligned} \tag{3.2.8}$$

(f) Reflector from **Mon** to **Grp**:

$$\begin{aligned}\mathbb{H} : \mathbf{Mon} &\to \mathbf{Grp} \\ M \mapsto \mathbb{H}M &= M \cup \{h^{-1} : h \in M\}\} \\ M \xrightarrow{m} N \mapsto \mathbb{H}M &\xrightarrow{\mathbb{H}m} \mathbb{H}N, x \mapsto \mathbb{H}m(x) = \begin{cases} m(x) & \text{if } x \in M \\ (m(x))^{-1} & \text{otherwise .}\end{cases}\end{aligned} \tag{3.2.9}$$

3.3 Adjunctions between $\mathbf{Mon}$ and $\mathbf{MSLat}$

In this part, we will consider the semilattice, not necessarily contains 0: $\mathbf{MSLat}$.
Consider two full $\mathbf{Mon}$'subagories $\mathbf{CMon}$, $\mathbf{CIMon}$ whose the objects are respectively the commutative monoids and the commutative monoids where every element is idempotent. We try to define $\mathbb{F}_1$ and $\mathbb{F}_2$ the left adjoints of the inclusion functors $\mathbb{I}_1 : \mathbf{CMon} \to \mathbf{Mon}$ and $\mathbb{I}_2 : \mathbf{CIMon} \to \mathbf{CMon}$.

3.3.1 Lemma. $\mathbf{CIMon} \cong \mathbf{MSLat}$.

Proof. We will not prove it in details but just state the ideas to get the functors isormorphic. For the one $\mathbf{MSLat}$ to $\mathbf{CIMon}$, $\mathbb{F}$ is just by remarking that the meet-semilattice $(S, \wedge)$ is a $\mathbf{CIMon}$-object by taking the meet operation as a binary operation. For the other one $\mathbf{CIMon}$ to $\mathbf{MSLat}$, $\mathbb{G}$ is just taking the binary operation as a meet operation. (We can think only on $\wedge$ by Remark 2.2.4). $\square$

For any monoid $(M, \cdot)$, define a binary relation $\mathcal{R}_M$ on M, such that for any $x, y \in M$,

$$(x, y) \in \mathcal{R}_M \quad \boldsymbol{iff} \quad \exists n \in \mathbb{N}, \sigma \in \mathcal{S}_n, a_1, \cdots, a_n \in M : x = \prod_{i=1}^{n} a_i, y = \prod_{i=1}^{n} a_{\sigma(i)},$$

where $\mathcal{S}_n$ is the group of permutation on $\{1, \cdots, n\}$.
It is an equivalence relation: the reflexivity is by the identity permutation, the symetricity is by each permutation being invertible, and the transitivity is by the fact that the composite of two permutations is also a permutation. So we have the quotient $CM = M/\mathcal{R}_M$.

Let $*$ be a binary operation on CM such that

$$\forall \bar{x}, \bar{y} \in CM : \quad \bar{x} * \bar{y} = \overline{xy}.$$

It is weel-defined and $(CM, *)$ is a commutative monoid.
For any commutative monoid $(C, \cdot)$, define a binary relation $\equiv_C$ on C such that for any $x, y \in C$, $x \equiv_C y$ if

(i) $(x = y)$ or $(x = aa, y = a)$ or $(x = a, y = aa)$ for some $a \in C$ or

(ii) x, y can be written $x = \prod_{i=1}^{i=n} a_i, y = \prod_{i=1}^{i=n} b_i$ with $a_i, b_i \in C$ and $a_i \equiv_C b_i$ for ecah $i = 1, \cdots, n$.

It is an equivalence relation and let's consider the quotient $IC = C/\equiv_C$.

Let $*$ be a binary operation on IC such that

$$\forall \bar{x}, \bar{y} \in IC : \quad \bar{x} * \bar{y} = \overline{xy}.$$

It is weel-defined and $(IC, *)$ is a commutative monoid whose every element is idempotent.
Now, we construct the functor $\mathbb{F}_1$ and $\mathbb{F}_2$, by the way

$$\begin{aligned} \mathbb{F}_1 : \mathbf{Mon} &\to \mathbf{CMon} \\ M &\mapsto \mathbb{F}_1(M) = CM \\ M \xrightarrow{f} N &\mapsto CM \xrightarrow{\mathbb{F}_1(f)} CN : x \mapsto f(x) \end{aligned} \qquad \begin{aligned} \mathbb{F}_2 : \mathbf{CMon} &\to \mathbf{CIMon} \\ C &\mapsto \mathbb{F}_2(C) = IC \\ C \xrightarrow{m} D &\mapsto IC \xrightarrow{\mathbb{F}_2(m)} ID : x \mapsto m(x) \end{aligned}$$

The $\mathbb{F}_1(f)$ and $\mathbb{F}_2(m)$ are well-defined because CN is commutative and every element of ID is idempotent. They are $\mathbf{Mon}$-morphisms then respectively $\mathbf{CMon}$-morphism and $\mathbf{CIMon}$-morphism. The functorialy are clear. Now, let's construct the natural transformations

$$\alpha := (\alpha_M : M \to \mathbb{I}_1\mathbb{F}_1 M, x \mapsto x)_{M \in \mathbf{Mon}} \quad \text{and} \quad \delta := (\delta_C : C \to \mathbb{I}_2\mathbb{F}_2 C, x \mapsto x)_{C \in \mathbf{CMon}}.$$

By definition of $\mathbb{F}_1(f)$ and $\mathbb{F}_2(m)$, we have $\alpha : Id_{\mathbf{Mon}} \overset{\bullet}{\Longrightarrow} \mathbb{I}_1\mathbb{F}_1$ and $\delta : Id_{\mathbf{CMon}} \overset{\bullet}{\Longrightarrow} \mathbb{I}_2\mathbb{F}_2$.
For any $M \xrightarrow{f} \mathbb{I}_1 C$ in $\mathbf{Mon}(M, \mathbb{I}_1 C)$, there is exactly one $\bar{f}$ in $\mathbf{CMon}(\mathbb{F}_1 M, C)$,$\bar{x} \mapsto \bar{f}(x) = f(x)$ so that $\mathbb{I}_1(\bar{f}) \circ \alpha_M = f$ and for any $C \xrightarrow{m} \mathbb{I}_2 I$ in $\mathbf{CMon}(C, \mathbb{I}_2 I)$, there is exactly one $\bar{m}$ in $\mathbf{CIMon}(\mathbb{F}_2 C, I)$,$\bar{x} \mapsto \bar{m}(x) = m(x)$ so that $\mathbb{I}_2(\bar{m}) \circ \delta_C = m$. The $\bar{f}, \bar{m}$ are well-defined because C is commutative and each element in I is idempotent. Thus $\mathbb{F}_1 \dashv \mathbb{I}_1$ and $\mathbb{F}_2 \dashv \mathbb{I}_2$.

Hence $\mathbf{CIMon}$ is reflective subcatory of $\mathbf{CMon}$ and $\mathbf{CMon}$ is reflective subcatory of $\mathbf{Mon}$.

3.3.2 Corollary. By Remark 2.7.9 with the free functor $\mathbb{M} : \mathbf{Set} \to \mathbf{Mon}$ in (3.2.5) and Lemma 3.3.1, We get the free functor $\mathbb{S} : \mathbf{Set} \to \mathbf{MSLat}$ by composing $\mathbb{F}_2\mathbb{F}_1\mathbb{M}$ (exactly $\mathbb{S} = \mathbb{G}\mathbb{F}_2\mathbb{F}_1\mathbb{M}$).

For remark, there is another way to construct the free semilattice in Picado and Pultr (2021). By Remark 3.1.6, we should have the isormophism between $\mathbb{S}S$ and the free semilattice gotten there $(\{X \subseteq S : X \text{ is finite}\}, \supseteq)$ for any set S. But here, we will show explicitly the isomorphism.

3.3.3 Proposition. For any set S, $\mathbb{S}S$ and $(\{X \subseteq S : X \text{ is finite}\}, \cup)$ are isomorphic in $\mathbf{MSLat}$.

Proof. The bijections are

$$f : \mathbb{S}S \to \{X \subseteq S : X \text{ is finite}\} \qquad g : \{X \subseteq S : X \text{ is finite}\} \to \mathbb{S}S$$
$$\prod_{i=1}^{i=n} s_i \mapsto \{s_i : i = 1, \cdots, n\} \qquad X \mapsto \prod_{s \in X} s$$

g is well-defined; for f : $\bar{x} = \bar{y}$ in $\mathbb{S}S$ means the elements of S appearing in x are the same of the elements of S appearing in y (ignoring the order: by $\mathcal{R}$, and repetition: by $\equiv$), then $f(\bar{x}) = g(\bar{y})$. Clearly, $f \circ g$, and $g \circ f$ are identity maps and these maps preserve the top. If C_x is the set of components of x, $C_{x \wedge y} = C_{xy} = C_x \cup C_y$. Then $f(x \wedge y) = f(x) \cup f(y)$ and $g(X \cup Y) = g(X) \wedge g(Y)$. Then f and g are isomorphisms of meet-semilattice. □

3.4 Construction free lattice

It is well-known, as shown in Johnstone (1982), that it is possible to construct the free lattice for any set. However, I will omit some details and present it differently. Theorem 3.1.5 guarantees the existence of a free functor respect to the forgetful functor $\mathbb{U} : \mathbf{Lat} \to \mathbf{Set}$. Here, I will emphasize the categorical aspects by stating the free functor arising in the free construction.

Let's consider a set X. We define the set RX as an extension of X such its elements are of form:

(i) $x \in X$ (we call such elements the generators) or

(ii) $\bigwedge S$ or $\bigvee S$ for some finite subset S of RX (where $\bigwedge\{x\} = \bigvee\{x\} = x$ for any $x \in RX$).

We could have an order relation $\leq$ where $(RX, \leq)$ is lattice and $\bigwedge$, $\bigvee$ are the meet and join in RX. We start by a relation $\mathcal{R}$ can be expressed recursively for any $u, v \in RX$, $u\mathcal{R}v$ if:

(a) $u = v$ or

(b) $u = \bigvee S$, $v = \bigvee R$ and for any $s \in S$, there is $r \in R : s\mathcal{R}r$ for S, R finite subsets of RX or

(c) $u = \bigwedge S$, $v = \bigwedge R$ and for any $r \in R$, there is $s \in S : s\mathcal{R}r$ for S, R finite subsets of RX or

(d) $u\mathcal{R}x$ and $x\mathcal{R}v$ for some $x \in RX$. (To make sure the transitivity).

We see that this relation is reflexive by (a) and it is transitive by (d). Now, let's consider the relation $\mathcal{E}$:

$$\mathcal{E} = \{(x, y) \in RX^2 : x\mathcal{R}y \text{ and } y\mathcal{R}x\}$$

Clearly, it is an equivalence relation and let's denote $FX = RX/\mathcal{E}$. In addition, we have an order relation $\leq$ on FX defined by $\bar{a} \leq \bar{b}$ if and only if $a\mathcal{R}b$. Obviously, the poset $(FX, \leq)$ has the top and bottom which are the class of $\bigwedge \emptyset$ and the class of $\bigvee \emptyset$. In the following, we abuse to confuse each element (in RX) and its class (in FX). By $a = \wedge\{a\}, b = \wedge\{b\}$ and (c), $\bigwedge\{a, b\}$ is a lower bound of $\{a, b\}$. Moreover, if $c \leq a$ and $c \leq b$, by $c = \wedge\{c\}$ and (c), $c \leq \bigwedge\{a, b\}$. Hence $\bigwedge\{a, b\}$ is the meet of a and b, $a \wedge b$. Similarly for the join, the join of a and b is $a \vee b = \bigvee\{a, b\}$. Thus $(FX, \leq)$ is a lattice. For any map $X \xrightarrow{f} Y$, we can define a new map recursively $FX \xrightarrow{Ff} FY$ such that $\bar{x} \mapsto \overline{f(x)}$ (for $x \in X$), $\overline{\bigwedge S} \mapsto \overline{\bigwedge f[S]}$ and $\overline{\bigvee S} \mapsto \overline{\bigvee f[S]}$. By construction of FX and FY, it is well-defined. Moreover, clearly, it preserves order, finite meet and finite join; then $\mathbf{Lat}$-morphism.

Now let's think about a functor from $\mathbf{Set}$ to $\mathbf{Lat}$,

$$\begin{aligned} \mathbb{L} : \mathbf{Set} &\to \mathbf{Lat} \\ X &\mapsto \mathbb{L}(X) = FX \\ X \xrightarrow{f} Y &\mapsto \mathbb{L}X \xrightarrow{\mathbb{L}f = Ff} \mathbb{L}Y. \end{aligned} \tag{3.4.1}$$

$\mathbb{L}$ respects the composition by the definition of Ff and Fg.
Now, let's construct a natural transformation $\eta : Id_{\mathbf{Set}} \xrightarrow{\bullet} \mathbb{UL}$ whose the components are

$$\eta_X : X \to \mathbb{UL}, \qquad x \mapsto \bar{x}.$$

We can say that for any lattice L and a map $X \xrightarrow{f} L$, there exists an unique $\mathbf{Lat}$-morphism g from FX to L satifying $\mathbb{U}g \circ \eta_X = f$. This g can be expressed recursively $\bar{x} \mapsto \overline{f(x)}$ (for $x \in X$), $\overline{\bigwedge S} \mapsto \overline{\bigwedge f[S]}$ and $\overline{\bigvee S} \mapsto \overline{\bigvee f[S]}$. Hence we see that $\mathbb{L}$ is the left adjoint to the forgetful functor $\mathbb{U} : \mathbf{Lat} \longrightarrow \mathbf{Set}$. The proof of almost all of the constructions basically follows from the fact all elements of FX can be expressed using finite meet and finite join with the element of x as the generators and conversely every word constructed using the lattice operations is in FX. We reduce the set of words generated by $X, \wedge$ and $\vee$, RX to FX so we can have the property of $\wedge, \vee$ in the lattice, like the idempotence of these operations.

3.5 Construction free frame

The downset functor is the classic way to get the free frame on set, it is seen in Picado and Pultr (2011). Recall that a subset A of semilattice L is called downset if for all $a \in A$, $\downarrow a = \{x \in l : x \leq a\} \subseteq A$.

3.5.1 Remark. Let $S \xrightarrow{f} R$ be a $\mathbf{MSLat}$-morphism:

(a) the set of all downsets of each semilattice is a frame where meet is intersection and join is union.

(b) for any dowset U, V, V_i of S, $\downarrow f[U \cap V] = \downarrow f[U] \cap \downarrow f[V]$ and $\downarrow f[\bigcup_{i \in I} V_i] = \bigcup_{i \in I} \downarrow f[V_i]$.

The downset functor $\mathbb{D}$ is defined as

$$\begin{aligned} \mathbb{D} : \mathbf{MSLat} &\to \mathbf{Frm} \\ S &\mapsto \mathbb{D}(S) = \{\downarrow X : X \subseteq S\} = \{U : U = \downarrow U\} \\ S \xrightarrow{f} R &\mapsto \mathbb{D}S \xrightarrow{\mathbb{D}f} \mathbb{D}R, \quad U \mapsto \downarrow f[U]. \end{aligned} \tag{3.5.1}$$

$\mathbb{D}$ is well-defined by Remark 3.5.1 , it remains for us to prove the functoriality. It is by the fact $\downarrow f[U] = \downarrow f[\downarrow U]$ for any f $\mathbf{MSLat}$-morphism. Nextly, define the component of $\lambda : Id_{\mathbf{MSLat}} \xrightarrow{\bullet} \mathbb{UD}$,

$$\lambda_S : S \to \mathbb{UD}S, \quad s \mapsto \lambda_S(s) = \downarrow s.$$

λ is obviously a natural transformation because λ_S is a $\mathbf{MSLat}$-morphism.

3.5.2 Lemma. Let S be a semilattice and L a be frame. For any $U, V \in \mathbb{D}S$, $S \xrightarrow{f} L$ in $\mathbf{MSLat}$,

$$\{f(c) : c \in U \cap V\} = \{f(a \wedge b) : a \in U, b \in V\}. \tag{3.5.2}$$

Proof. From left to right, take $a = b = c$. From right to left, take $c = a \wedge b \in U \cap V$ since $U, V \in \mathbb{D}S$. □

3.5.3 Lemma. Let S be a semilattice and L a be frame. For any $S \xrightarrow{f} \mathbb{U}L$ in $\mathbf{MSLat}$,

$$\exists! g \in \mathbf{Frm}(\mathbb{D}S, L) : \mathbb{U}g \circ \lambda_S = f.$$

Proof. We take $g(U) = \bigvee f[U]$. Obviously, it preserves all joins. For the finite meet, we use (3.5.2),

$$g[U] \wedge g[V] = (\bigvee f[U]) \wedge (\bigvee f[V]) = \bigvee \{f(a) \wedge f(b) : a \in U, b \in V\} \overset{(3.5.2)}{=} g[U \cap V].$$

So, g is $\mathbf{Frm}$-morphism. Moreover, it is clear that for any $s \in S$, $\mathbb{U}g \circ \lambda_L(s) = \bigvee f[\downarrow s] = f(s)$. Hence, the existence. Now, let $h \in \mathbf{Frm}(\mathbb{D}S, L)$ such that $\mathbb{U}h \circ \lambda_L = f$. Then for any $U \in \mathbb{D}S$: $h(U) = h(\bigcup_{s \in U} \downarrow s) = \bigvee_{s \in U} h(\downarrow s) = \bigvee_{s \in U} f(s) = \bigvee f[U] = g(U)$. Thus, the unicity. □

3.5.4 Proposition. We conclude that $\mathbb{D}$ is the left adjoint to the forgetful functor $\mathbb{U} : \mathbf{Frm} \to \mathbf{MSLat}$.

3.5.5 Corollary. Finally, by Remark 2.7.9 and Corollary 3.3.2, the composing $\mathbb{DS}$ is the free functor respect to the forgetful $\mathbf{Frm} \to \mathbf{Set}$.

3.6 Construction freely generated frame from a site.

In this section, the main definitions and concepts are from Johnstone (1982). But I will provide another proofs using simpler definitions, presenting Lemma 3.6.7 and Remark 3.6.9, which allow us to obtain explicitly the unique factorization of a $\mathbf{Frm}$-morphism claimed by Johnstone (1982) on page 59. Here, we will consider the site as a category and emphasize the adjunctions between this category and $\mathbf{Frm}$.

3.6.1 Definition (Johnstone (1982) p57)**.** Given a meet-semilattice A, a **coverage** on A is an element $\mathcal{C}$ of $\Big(\mathcal{P}(\mathcal{P}(A))\Big)^A$ such that for any $a \in A$, $\mathcal{C}(a) \in \mathcal{P}(\mathcal{P}(\downarrow a))$ satisfying the **meet-stability** property:

$$\forall a, b \in A : \quad b \leq a \implies \forall S \in \mathcal{C}(a) : \{b \wedge s : s \in S\} \in \mathcal{C}(b). \tag{3.6.1}$$

We construct a category $\mathbf{Sit}$ whose the objects are all sites, pair $(A, \mathcal{C})$ where A is meet-semilattice and $\mathcal{C}$ is coverage on A and the morphisms from $(A, \mathcal{C})$ to $(A', \mathcal{C}')$ are all $\mathbf{SLat}$-morphisms f from A to A' such that:

$$\forall a \in A : \quad \forall S \in \mathcal{C}(a), f[S] \in \mathcal{C}'(f(a)). \tag{3.6.2}$$

We need to check composition but it is trivial. Now, define

$$\begin{aligned} \mathbb{V}: \mathbf{Frm} &\to \mathbf{Sit} \\ L \mapsto \mathbb{V}(L) &= (L, \mathcal{C}_L) \quad \text{where } \mathcal{C}_L : x \mapsto \{S \subseteq L : \bigvee S = x\} \\ L \xrightarrow{f} M &\mapsto \mathbb{V}L \xrightarrow{\mathbb{V}f} \mathbb{V}N, x \mapsto f(x). \end{aligned} \tag{3.6.3}$$

It is well-defined because for all L frame, $\mathcal{C}_L$ is a coverage on L and for all $L \xrightarrow{f} M$ **Frm**-morphism, for any $S \subseteq L$: $\bigvee S = a$ implies $\bigvee f[S] = f(a)$ which mean that $\mathbb{V}(L) = (L, \mathcal{C}_L) \in \mathbf{Sit}$ and $\mathbb{V}(f)$ ***in*** **Sit**. The condition of the fonctuariality is obvious.

3.6.2 Definition (Johnstone (1982) p58)**.** Given a site $(A, \mathcal{C})$, a downset I of A is said $\mathcal{C}-$ideal (or simply **Sit**-ideal) if

$$\forall a \in A: \quad (\exists R \in \mathcal{C}(a), R \subseteq I) \implies a \in I. \tag{3.6.4}$$

3.6.3 Lemma. The set of all $\mathcal{C}$-ideals of A, $\mathcal{C}$-**Idl**(A) equipped with the inclusion is a frame.

Johnstone (1982) proved it using nucleus. Here, we do another one but still use Proposition 2.2.18. Then, It suffices to prove that $\mathcal{C}$-**Idl**(A) is a sublocale of the locale $\mathbb{D}A$ seen at section 3.5.

Proof. It is obvious that any intersection of $\mathcal{C}$-ideal is $\mathcal{C}$-ideal.
Now, let $I \in \mathcal{C}$-**Idl**(A) and $D \in \mathbb{D}A$. It remains to show that $H = D \to I \in \mathcal{C}$-**Idl**$(A)$. Explicitly:

$$H = \bigcup \{X \in \mathbb{D}A : X \cap D \subseteq I\}.$$

Let $a \in A$ and $S \in \mathcal{C}(a)$ such that $S \subseteq H$. We should have $(\downarrow a) \cap (D \setminus I) = \emptyset$. Indeed, otherwise, there is $b \leq a$ such that $b \in D$ and $b \notin I$. By meet-stability and (3.6.4), we have a coverage of b, T

$$T := \{b \wedge s : s \in S\} \nsubseteq I \implies \exists t \in T : t \notin I \implies \exists t \in T, t \leq b : t \notin I \implies \exists t \in T, t \in D : t \notin I.$$

Since $t \in T$, there is some $s \in S$ such that $t = s \wedge b$. Then $t \leq s$.
Take an abitrary $X \in \mathbb{D}A$ which contains s. Then $t \in X$. So $t \in X \cap D$ but $t \notin I$. Then $X \cap D \nsubseteq I$. Hence there is $s \in S$ such that for any $X \in \mathbb{D}A$, $s \in X$ implies $X \cap D \nsubseteq I$. That contradicts $S \subseteq H$. Thus, we must have $(\downarrow a) \cap (D \setminus I) = \emptyset$. Then, taking $Y = \downarrow a \in \mathbb{D}A$, $Y \cap D \subseteq I$. So $a \in H$. □

In the next, we use the notation $\langle E \rangle$ for the smallest ideal of the corresponding site, containing E.

3.6.4 Remark. Clearly, the join in $\mathcal{C}$-**Idl**(A) is : $\bigsqcup_{i \in E} I_i := \langle \bigcup_{i \in E} I_i \rangle = \bigcap \{I \in \mathcal{C}\text{-}\mathbf{Idl}(A) : \bigcup_{i \in E} \subseteq I\}$.

3.6.5 Proposition. Given a site $(A, \mathcal{C})$. Let $u : A \to \mathcal{C}$-**Idl**(A) $a \mapsto u(a) = \langle \boldsymbol{a} \rangle$. Explicitly

$$u(a) = \bigcap \{I \in \mathcal{C}\text{-}\mathbf{Idl}(A) : \downarrow a \subseteq I\} = \langle \downarrow \boldsymbol{a} \rangle = \bigcap \{I \in \mathcal{C}\text{-}\mathbf{Idl}(A) : a \in I\} = \langle \boldsymbol{a} \rangle. \tag{3.6.5}$$

It is well-defined as a map. Moreover it is a **Sit**-morphism if we equip $\mathcal{C}$-**Idl**(A), $\mathcal{C}_{\mathcal{C}\text{-}\mathbf{Idl}(A)}$.

It is clear that $u(a)$ is the smallest $\mathcal{C}$-**Idl**(A) containning $\downarrow a$ (or only a).
We will write this map u_A to specify that we work in $(A, \mathcal{C})$ (by abuse, we let $\mathcal{C}$ but not forget).

Proof. It is well-defined because the arbitrary intersection of $\mathcal{C}$-ideal is an $\mathcal{C}$-ideal and $u(1) = A$.

On the other hand, $\downarrow(a\wedge b)\subseteq(\downarrow a)$ then for any $I\in\mathcal{C}\text{-}\mathbf{Idl}(A)$, $\downarrow a\subseteq I$ implies $\downarrow(a\wedge b)\subseteq I$. Then

$$\{I\in\mathcal{C}\text{-}\mathbf{Idl}(A):\downarrow a\subseteq I\}\subseteq\{I\in\mathcal{C}\text{-}\mathbf{Idl}(A):\downarrow(a\wedge b)\subseteq I\}.$$

So $u(a\wedge b)\subseteq u(a)$. Similar for b, we get $u(a\wedge b)\subseteq u(a)\cap u(b)$. To prove the converse, let's denote

$$J=\{x\in A:\forall s\leq a, x\wedge s\in u(a\wedge b)\}\quad\text{and}\quad K=\{x\in A:\forall t\in J, x\wedge t\in u(a\wedge b)\}.$$

It is clear that $b\in J$, $a\in K$ and $J\cap K\subseteq u(a\wedge b)$.

Let $y\in A, R\in\mathcal{C}(y)$ and $R\subseteq J$. Consider an arbitrary $s\leq a$. By the meet-stability property,

$$R_s=\{t\wedge y\wedge s=t\wedge s:t\in R\}\in\mathcal{C}(y\wedge s)$$

and if $z\in R_s$, $z=t\wedge s$ for some $t\in R\subseteq J$ then $z\in u(a\wedge b)$. So $R_s\subseteq u(a\wedge b)$ as $s\leq a$. Since, $u(a\wedge b)$ is a $\mathcal{C}$-ideal, $y\wedge s\in u(a\wedge b)$. As we have taken an s arbitrary, we have $y\in J$, and hence J is a $\mathcal{C}$-ideal of A (easy to check it is downset) containing b. Thus $u(b)\subseteq J$.

Let $y\in A, S\in\mathcal{C}(y)$ and $S\subseteq K$. Consider an arbitrary $s\in J$. By the meet-stability property,

$$S_s=\{t\wedge y\wedge s=t\wedge s:t\in S\}\in\mathcal{C}(y\wedge s)$$

and $S_s\subseteq\{t\wedge s:t\in K\}\subseteq u(a\wedge b)$. Since, $u(a\wedge b)$ is a $\mathcal{C}$-ideal, $y\wedge s\in u(a\wedge b)$. As we have taken an s arbitrary, we have $y\in K$ and hence K is a $\mathcal{C}$-ideal of A (it as downset) containing a. Thus $u(a)\subseteq K$.

Therefore $u(a)\cap u(b)\subseteq J\cap K\subseteq u(a\wedge b)$. Hence the equality and finally u is a $\mathbf{SLat}$-morphism.

Nextly, let $a\in A$ and $S\in\mathcal{C}(a)$. Clearly $S\subseteq\bigcup_{s\in S}u(s)\subseteq\bigsqcup u[S]$. Then $a\in\bigsqcup u[S]$ (since it is a $\mathcal{C}\text{-}\mathbf{Idl}(A)$). Then $u(a)\subseteq\bigsqcup u[S]$ (By $u(a)$ is the smallest $\mathcal{C}$-ideal containning a). Moreover if $y\in u[S]$, there is some $s\in S$ such that $y=u(s)\subseteq u(a)$ (by $S\subseteq\downarrow a$). So $u(a)$ is an upper bound of $u[S]$. Then $\bigsqcup u[S]\subseteq u(a)$. So $\bigsqcup u[S]=u(a)$ and $u[S]\in\mathcal{C}_{\mathcal{C}\text{-}\mathbf{Idl}(A)}(u(a))$. Hence u is a $\mathbf{Sit}$-morphism. □

3.6.6 Proposition (Johnstone (1982) p58)**.** For any downset U and V of A, $\langle U\cap V\rangle=\langle U\rangle\cap\langle V\rangle$.

Proof. Similar to the above proof. The first inclusion is by $U\cap V\subseteq\langle U\rangle\cap\langle V\rangle$. For the another one, we just change $J=\{x\in A:\forall s\in U, x\wedge s\in\langle U\cap V\rangle\}$ and $K=\{x\in A:\forall s\in J, x\wedge s\in\langle U\cap V\rangle\}$. □

3.6.7 Lemma. Let $f:(A,\mathcal{C})\longrightarrow(B,\mathcal{B})$ be a $\mathbf{Sit}$-morphism.

(a) If J is a $\mathcal{B}$-ideal of B then $f^{-1}[J]$ is $\mathcal{C}$-ideal of A.

(b) For any X subset of A and K $\mathcal{B}$-ideal of B: $f[X]\subseteq K$ if and only if $f[\langle X\rangle]\subseteq K$.

(c) In particular, $\langle f[\langle X\rangle]\rangle=\langle f[X]\rangle$.

(d) For any $(X_i)_{i\in E}$ subsets of A (or B), K $\mathbf{Sit}$-ideal: $\bigcup_{i\in E}X_i\subseteq K$ if and only if $\bigcup_{i\in E}\langle X_i\rangle\subseteq K$.

(e) In particular, $\langle\bigcup_{i\in E}\langle X_i\rangle\rangle=\langle\bigcup_{i\in E}X_i\rangle$.

Proof. For (a), assume that there is $R\in\mathcal{C}(x)$ and $R\subseteq f^{-1}[J]$. As f is $\mathbf{Sit}$-morphism $f[R]\in\mathcal{B}(f(x))$. Moreover $f[R]\subseteq f[f^{-1}[J]]\subseteq J$. Hence $f(x)\in J$ and $x\in f^{-1}[J]$.
For (b), let K an $\mathcal{B}$-ideal such that $f[X]\subseteq K$. We have $X\subseteq f^{-1}[f[X]]\subseteq f^{-1}[K]$, which is an $\mathcal{C}$-ideal by (a). Then $\langle X\rangle\subseteq f^{-1}[K]$. So $f[\langle X\rangle]\subseteq f[f^{-1}[K]]\subseteq K$. The converse is trivial.
For (d), let K and $\mathbf{Sit}$-ideal such that $\bigcup_{i\in E}X_i\subseteq K$. We have $X_i\subseteq K$ for each $i\in E$. Then $\langle X_i\rangle\subseteq K$ for each $i\in E$. So $\bigcup_{i\in E}\langle X_i\rangle\subseteq K$. The converse is trivial. □

Now let's construct the functor $\mathbb{F} : \mathbf{Sit} \longrightarrow \mathbf{Frm}$:

$$\begin{aligned} \mathbb{F} : \mathbf{Sit} &\to \mathbf{Frm} \\ (A, \mathcal{C}) &\mapsto \mathbb{F}(A, \mathcal{C}) = \mathcal{C}\text{-}\mathbf{Idl}(A) \\ (A, \mathcal{C}) \xrightarrow{f} (B, \mathcal{B}) &\mapsto \mathcal{C}\text{-}\mathbf{Idl}(A) \xrightarrow{\mathbb{F}f} \mathcal{B}\text{-}\mathbf{Idl}(B), I \mapsto \langle f[I] \rangle. \end{aligned} \tag{3.6.6}$$

Remark 3.5.1 (b) and Proposition 3.6.6 ensures us that $\mathbb{F}(f)$ preserves meet. Lemma 3.6.7 (c) and (e) ensures that $\mathbb{F}(f)$ preserves all joins. Combining these with Lemma 3.6.3, $\mathbb{F}$ is well-defined.
Lemma 3.6.7 (c) ensures us that it respects the composition. Hence, the functoriality of $\mathbb{F}$.

3.6.8 Proposition. We have a natural transformation $u : Id_{\mathbf{Sit}} \overset{\bullet}{\Longrightarrow} \mathbb{VF}$ of component $(u_A)_{(A,\mathcal{C}) \in \mathbf{Sit}}$.

Proof. Let $A \xrightarrow{f} B$ a $\mathbf{Sit}$-morphism. For any $x \in A$, we apply Lemma 3.6.7 (c) for $X = \{x\}$ to get: $u_B \circ f(x) = \langle f(x) \rangle = \langle f(\{x\}) \rangle = \langle f(\langle \{x\} \rangle) \rangle = \langle f(u_A(x)) \rangle = \mathbb{VF}f \circ u_A(x)$. □

3.6.9 Remark. For any frame B and $\mathbf{Sit}$-morphism $(A, \mathcal{C}) \xrightarrow{f} \mathbb{V}B = (B, \mathcal{C}_B)$:

(a) in $\mathbb{V}(B) = (B, \mathcal{C}_B)$, $\downarrow b$ is $\mathcal{C}_B$-ideal for each $b \in B$. In particular $\langle b \rangle = \downarrow b$.

(b) consequently, $\bigvee f[\langle a \rangle] = f(a)$ for each $a \in A$.

Proof. For (a): let $y \in B$ and $R \in \mathcal{C}_B(y)$ (i.e. $\bigvee R = y$) and $R \subseteq \downarrow b$. So, $y = \bigvee R \leq \bigvee \downarrow b = b$ then $y \in \downarrow b$. Hence $\downarrow b$ is a $\mathcal{C}_B$-ideal. For (b): since $f[\{a\}] \subseteq \downarrow f(a)$, (a) and Lemma 3.6.7 (b) tell us $f[\langle a \rangle] \subseteq \downarrow f(a)$. Then, $\bigvee f[\langle a \rangle] \leq f(a)$. By $f(a) \in f[\langle a \rangle], \bigvee f[\langle a \rangle] = f(a)$. □

3.6.10 Proposition. For any site $(A, \mathcal{C})$ and frame B, $\mathbf{Sit}$-morphism $f : (A, \mathcal{C}) \longrightarrow \mathbb{V}(B) = (B, \mathcal{C}_B)$, there is an unique frame homorphism $g : \mathbb{F}(A, \mathcal{C}) = \mathcal{C}\text{-}\mathbf{Idl}(A) \longrightarrow B$ such that $\mathbb{V}g \circ u_A = f$.

Proof. The proof is similar to Lemma 3.5.3. We take $g(I) = \bigvee f[I]$. The meet in $\mathcal{C}\text{-}\mathbf{Idl}(A)$ is the same as in $\mathbb{D}A$ but the joins are different in general. Let $(I_i)_{i \in E}$ a collection of $\mathcal{C}\text{-}\mathbf{Idl}(A)$ of A. We have $f[\bigcup_{i \in E} I_i] \subseteq \downarrow \bigvee f[\bigcup_{i \in E} I_i]$. By Lemma 3.6.7 (b) with Remark 3.6.9 (a), $f[\bigsqcup_{i \in E} I_i] \subseteq \downarrow \bigvee f[\bigcup_{i \in E} I_i]$. So $\bigvee f[\bigsqcup_{i \in E} I_i] \leq \bigvee f[\bigcup_{i \in E} I_i]$. Then $\bigvee f[\bigsqcup_{i \in E} I_i] = \bigvee f[\bigcup_{i \in E} I_i]$ (because the converse is trivial). Thus $g[\bigsqcup_{i \in E} I_i] = \bigvee_{i \in E} \left(\bigvee f[I_i] \right) = \bigvee_{i \in E} g[I_i]$. Hence, g is $\mathbf{Frm}$-morphism. Using Remark 3.6.9 (b), for any $a \in A$, $\mathbb{V}g \circ \mu_A(a) = \bigvee f[\langle a \rangle] = f(a)$. Thus, the existence.

Now, let $h \in \mathbf{Frm}(\mathcal{C}\text{-}\mathbf{Idl}(A), B)$ such that $\mathbb{V}h \circ \mu_A = f$. Then for any $I \in \mathcal{C}\text{-}\mathbf{Idl}(A)$:
$h(I) = h(\bigsqcup_{x \in I} \langle x \rangle) = \bigvee_{x \in I} h(\langle x \rangle) = \bigvee_{x \in I} f(x) = \bigvee f[I] = g(I)$. Hence, the uniqueness. □

3.6.11 Corollary. Combining Propositions 3.6.8 and 3.6.10, we conclude that $\mathbb{F}$ in (3.6.6) is the left adjoint to $\mathbb{V}$ in (3.6.3) ($\mathbb{F} \dashv \mathbb{V}$).

Johnstone (1982) defined "frame freely generated by a site $(A, \mathcal{C})$" by a frame L with an existence of a $\mathbf{SLat}$-morphism $\mu : A \to L$ which "transforms covers to joins" and is universal among such maps. By remarking that "transforms covers to joins" means being a $\mathbf{Sit}$-morphism if we equip L by $\mathcal{C}_L$. We should be allowed to say that a frame freely generated by any site $(A, \mathcal{C})$ exists (by Johnstone (1982) $\mathcal{C}\text{-}\mathbf{Idl}(A)$) and category theory, Remark 3.1.6[3] ensures the uniqueness (up to isomorphism).

[3]Remark 3.1.6 talk about free functor but it is still true for any left adjunction. It is seen by folowing the proof.

3.7 Adjunctions between MSLat and Sit

Here, let's denote particularly the forgetful $\mathbb{W} : \mathbf{Sit} \to \mathbf{MSLat}$. We try to see the interactions between $\mathbf{MSLat}$ and $\mathbf{Sit}$ to check if the concept of frame "freely generated by a site" has something to do with the downset functor $\mathbb{D}$ in 3.5. Let's start with a lemma.

3.7.1 Lemma. For any meet-semilattice A, the following maps are coverages on A:

$$\begin{aligned} \mathcal{E}_A : A &\to \mathcal{P}(\mathcal{P}(A)) \\ a &\mapsto \mathcal{E}_A(a) = \emptyset; \end{aligned} \qquad \begin{aligned} d_A : A &\to \mathcal{P}(\mathcal{P}(A)) \\ a &\mapsto d_A(a) = \mathcal{P}(\downarrow a). \end{aligned}$$

Proof. Firstly they are well-defined and the elements of $\mathcal{E}_A(a)$ and $d_A(a)$ are all subsets of $\downarrow a$. Moreover, it is no S so $S \in \mathcal{E}_A(a)$ for ecah $a \in A$ and for any S subset of A, $\{s \wedge b : s \in S\} \in \mathcal{P}(\downarrow b)$. $\square$

Now, we define two functors

$$\begin{aligned} \mathbb{L} : \mathbf{MSLat} &\to \mathbf{Sit} \\ A &\mapsto \mathbb{L}(A) = (A, \mathcal{E}_A) \\ A \xrightarrow{f} B &\mapsto \mathbb{L}A \xrightarrow{\mathbb{L}(f)} \mathbb{L}B : x \mapsto f(x); \end{aligned} \qquad \begin{aligned} \mathbb{T} : \mathbf{MSLat} &\to \mathbf{Sit} \\ A &\mapsto \mathbb{T}(A) = (A, d_A) \\ A \xrightarrow{f} B &\mapsto \mathbb{T}A \xrightarrow{\mathbb{T}(f)} \mathbb{T}B : x \mapsto f(x). \end{aligned}$$

The object mapping and morphism mapping are well-defined and the compositions are gotten trivially. Nextly, we define two natural transformations $\Sigma : Id_{\mathbf{MSLat}} \overset{\bullet}{\Longrightarrow} \mathbb{W}\mathbb{L}$ and $\Lambda : Id_{\mathbf{Sit}} \overset{\bullet}{\Longrightarrow} \mathbb{T}\mathbb{W}$ such theirs compenents are $x \mapsto x$. They are clear to be $\mathbf{SMLat}$-morphism and $\mathbf{Sit}$-morphism. The naturalities are trivial. For the universal property, we notice a few observations that seem very obvious.

3.7.2 Remark. Let A be a semilattice and $(S, \mathcal{C})$ be a site. For any $A \xrightarrow{f} S$ in $\mathbf{MSLat}(A, S)$, the map $(A, \mathcal{E}_A) \xrightarrow{g} (S, \mathcal{C}), x \mapsto f(x)$ is the only $\mathbf{Sit}$-morphism that satifies $\mathbb{W}(g) \circ \Sigma_A = f$.

3.7.3 Remark. Let $(S, \mathcal{C})$ be a site and A be a semilattice. For any $(S, \mathcal{C}) \xrightarrow{f} (A, d_A)$ in $\mathbf{Sit}((S, \mathcal{C}), (A, d_A))$, the map $S \xrightarrow{g} A, x \mapsto f(x)$ is the only $\mathbf{MSLat}$-morphism that satifies $\mathbb{T}(g) \circ \Lambda_{(S,\mathcal{C})} = f$.

3.7.4 Proposition. We conclude that $\mathbb{L}$ and $\mathbb{T}$ are the left and right adjoint to the forgetful functor $\mathbb{W}$.

This is another another ways to get the downset functor in section 3.5, $\mathbb{D} = \mathbb{F}\mathbb{L}$, $\mathbb{F}$ be the one in 3.6.6. Then, we should check that $\mathcal{E}_A$-**Idl**$(A) = \mathbb{D}A$ but it is clear.

3.8 Free Heyting algebra

We give the free Heyting algebra with one generator. Let's consider the set $X = \{(1, 0)\}$.
Let's define FX and we will try to explain why it is the free Heyting algebra on X.

$$FX := \{(x, y) \in \mathbb{N} \times \mathbb{N} : y \leq x + 1 \leq y + 3\} \cup \{\infty = (+\infty, +\infty)\}.$$

The order is defined by $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Clearly, it is a bounded lattice.

We could easily verify that if $x_1 > x_2$ and $y_1 > y_2$, $(x_1, y_1) \to (x_2, y_2) = (x_2, y_2)$; if $x_1 > x_2$ and $y_1 \leq y_2$, $(x_1, y_1) \to (x_2, y_2) = (x_2, x_2 + 1)$; if $x_1 \leq x_2$ and $y_1 > y_2$, $(x_1, y_1) \to (x_2, y_2) = (y_2 + 2, y_2)$ and otherwise $(x_1, y_1) \to (x_2, y_2) = \infty$. Hence FX is an Heyting algebra.

We can also see that every element of FX can be gotten recursively using only $(1,0),(0,0),\infty,\wedge,\vee,\to$. Then for every Heyting Algebra H and for every map $X \xrightarrow{f} H$ there exists an unique $\mathbf{Hey}$-morphism g from FX to H such that $g(1,0) = f(1,0)$. That was proven at Balbes and Dwinger (1977). This g can be expressed recursively as $(1,0) \mapsto f(1,0)$, $(0,0) \mapsto 0$, $\infty \mapsto 1$, $a \wedge b \mapsto g(a) \wedge g(b)$, $a \vee b \mapsto g(a) \vee g(b)$ and $a \to b \mapsto g(a) \to g(b)$. Thus FX is the free Heyting algebra generated by X.

At Johnstone (1982), Johnstone claimed that the free complete Heyting algebra respect to the set with one element is $FX \cup \{*\}$. That shows a big difference between the category of frames and the category of complete Heyting algebras even they have same objects. By Corollary 3.5.5, the free frame generated by a set with one element $\{e\}$ is isomorphic to $\{0, e, 1\}$.

3.9 Some examples of forgetful functor no free functor

For some categories, we will conclude the non-existence of free functor using LAPC and RAPL at Theorem 2.7.10 and Theorem 3.1.5. The no-existence of a free functor tells us just that there is some object that has no free object but perhaps some other object has a free object. For instance for the case of $\mathbf{CLat}$, we have a free complete lattice one generator but it has no free functor.

3.9.1 $\mathbf{Field}$: category of fields whose the morphisms are the $\mathbf{Ring}$-morphisms.

Recall the characteristic of field K is the smallest $k \in \mathbb{N}^*$ such that $k \cdot 1 = 1 + 1 + \cdots + 1 = 0$ (k times) if there is such k but otherwise it is zero. We know that the characteristic of any field is zero or prime number.

3.9.2 Proposition. For any fields K, L: if there is a $\mathbf{Field}$-morphism $f : K \longrightarrow L$ then K and L have a same characteristic. In particular, $\mathbf{Field}$ has neither an initial object nor a final object.

Proof. Let $f : K \longrightarrow L$ be a $\mathbf{Field}$-morphism. For the case where the characteristic of K is a prime k. We have $1_L + 1_L + \cdots + 1_L = f(1_K + 1_K + \cdots + 1_K) = f(0_K) = 0_L$ (all is k times). Because k is prime, it must be the smallest satisfying $1_L + 1_L + \cdots + 1_L = 0$ (k times), and then the characteristic of L is also k. We can use the fact that the kernel of f is an ideal of K, which is (0) or (K) to conclude that f is injective. It is by $f(1_K) = 1_L$ and then the kernel of f must be (0) not K. Now, we use it to prove that if K has a characteristic zero then L has a characteristic zero. Inded, characteristic of K is zero means for every $k \in \mathbb{N}^*$, $k \cdot 1_K \neq 0_K$ which implies $k \cdot 1_L = k \cdot f(1_K) = f(k \cdot 1_L) = f(0_K) \neq 0_L$. $\square$

If the forgetful $\mathbb{U} : \mathbf{Field} \longrightarrow \mathbf{Set}$ has a left adjoint $\mathbb{F} : \mathbf{Set} \to \mathbf{Field}$ then this $\mathbb{F}$ preserves colimits. As $\emptyset$ is an initial object in $\mathbf{Set}$ and colimit of the diagram $\mathbb{E} : 0 \to \mathbf{Set}$. Then $\mathbb{F}\emptyset$ is the colimit of the diagram $\mathbb{F}\mathbb{E} : 0 \to \mathbf{Field}$. Hence $\mathbb{F}\emptyset$ is an initial object in $\mathbf{Field}$ which contradicts the Proposition above.

3.9.3 $\mathbf{CLat}$: category of complete lattice

In Johnstone (1982), it is proven at page 31, that the complete lattice generated by $X = \{x, y, z\}$ does not exist. Hence, Theorem 3.1.5 ensures us that the forgetful $\mathbb{U} : \mathbf{Clat} \longrightarrow \mathbf{Set}$ has no left adjoint (free functor respect to $\mathbb{U}$).

4. Final Remarks

The categorical approach to free objects in a category allows us to conclude that there are no free objects when the relevant forgetful functor does not preserve some limits using **RAPL**. The General Adjoint Functor Theorem on the other hand might help us conclude if the free functors exist.

Limits and colimits in $\mathbf{Set}$ are familiar and easy to construct. In instances where we have free objects and the concrete[1] category is complete, we can use **RAPL**: the underlying set of the products in $\mathbf{Grp}$ ($\mathbf{Top}$, $\mathbf{Frm}$, $\mathbf{Lat}$, etc.) is the product of the underlying sets, so we can work with the fact that the projections are morphisms to get the product in each category. It is similar for all limits.

In other words due to the fact that $\mathbf{Grp}$ ($\mathbf{Top}$, $\mathbf{Frm}$, $\mathbf{Lat}$, etc.) admit free functors, the forgetful functors, being right adjoints, must preserve limits. The same holds for the existing products in $\mathbf{Met}_u$.

On the other hand, using **LAPC**, if a concrete category admits a free functor, then the coproduct of free objects is the free object on the coproduct of sets. For instance, the free group on the set $X \coprod Y$ is the free product (coproduct) of the free groups on sets X and Y (see Corollary 4.5.7 in Riehl (2017)).

Using the same reasoning, we can also conclude that products and equalizers[2] in $\mathbf{Ab}$ are none other than those in $\mathbf{Grp}$, and every existing product in $\mathbf{CMet}_u$ is none other than that in $\mathbf{Met}_u$. This reasoning is valid for any reflective subcategory.

Definitions and tools from category theory help us study the free objects in a number of contexts. An interesting investigation is that of complete Heyting algebras. We have seen that free complete Heyting algebras differ from free frames even though they are the same algebraic objects. The difference arises because the morphisms of the relevant categories differ, underlying the crucial role of morphisms in understanding mathematical structures – exactly the point of departure for category theory. For the free objects in $\mathbf{Hey}$, there are many works discussing finitely generated cases. Regarding $\mathbf{CHey}$, constructing free objects or proving their non-existence is an intriguing (and very challenging) open research question that would extend this project.

[1] A concrete category is a category resembling a category of sets equipped with structures.
[2] We assume that $\mathbf{Ab}$ is complete.

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